

Functional Analysis in  
Interdisciplinary  
Applications 2023

International  
Mathematical  
Conference

Antalya, Türkiye  
October 02 - 07, 2023

**ABSTRACT  
BOOK**

International Mathematical  
Conference

“Functional Analysis in  
Interdisciplinary Applications”

# ABSTRACT BOOK

of the conference FAIA2023

Edited by

Allaberen Ashyralyyev,

Michael Ruzhansky,

Makhmud Sadybekov

October 02 – October 07, 2023

Bahçeşehir University,  
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**Edited by Allaberen Ashyralyev, Michael Ruzhansky, Makhmud Sadybekov**

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Bahçeşehir University (Türkiye), Analysis & PDE Center, Ghent University (Belgium), and Institute of Mathematics and Mathematical Modeling (Kazakhstan), being the organizers of the conference, present this Abstract book, which contains brief abstracts of the reports of the participants of the International Mathematical Conference “Functional Analysis in Interdisciplinary Applications” (FAIA2023). This Conference aims to bring together mathematicians working in the functional analysis and its applications.

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## FOREWORD

On behalf of the Organizing Committee, we are pleased to invite you to the International Mathematical Conference “Functional Analysis in Interdisciplinary Applications” (FAIA2023).

This onference aims to bring together mathematicians working in the functional analysis and its applications.

The meeting will be held on October 02–October 07, 2023 in Antalya, Türkiye. The conference will consist of plenary lectures, and sectional oral presentations.

This Astract book contains brief abstracts of the reports of the participants of the FAIA2023. The collection of abstracts is organized in alphabetical order by the last name of the first author.

We would like to thank our main sponsors Bahçeşehir University, Türkiye, Institute of Mathematics and Mathematical Modeling, Kazakhstan, and Ghent Analysis & PDE Center, Belgium. We also would like to thank to all participants and Technical Program Committee Members.

With our best wishes and warm regards,

**Prof. Allaberen Ashyralyev**

**Prof. Michael Ruzhansky**

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## Direct and Inverse Problems For a Nonlinear Parabolic-Hyperbolic Equation

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This work is devoted to the study of direct and inverse problems with a nonlinear gluing condition in a mixed parabolic-hyperbolic domain with a characteristic triangle in the hyperbolic part. It should be noted that most of the problems studied were considered in rectangular domains [1], [2], but in characteristic hyperbolic domain they have not previously been studied.

We consider

$$\begin{cases} u_{xx} - {}_C D_{oy}^\alpha u + \beta u + \gamma u^2 + p_1(x, y, u(x, +0)) = f(x)g_1(y) & y > 0 \\ u_{xx} - u_{yy} + p_2(x, y, u(x, -0)) = f(x)g_2(y), & y < 0 \end{cases} \quad (1)$$

in domain  $\Omega = \{(x, y) : 0 < x < l; 0 < t < h\}$ , restricted with segments:  $A_1A_2 = \{(x, y) : x = l, 0 < y < h\}$ ,  $B_1B_2 = \{(x, y) : x = 0, 0 < y < h\}$ ,  $B_2A_2 = \{(x, y) : y = h, 0 < x < l\}$  for  $y > 0$  and by the characteristics  $B_1 : x + y = 0$ ,  $A_1 : x - y = l$  of equations (1) at  $y < 0$ , where  $g_1(y)$ ,  $g_2(y)$  are given sufficiently smooth function,  ${}_C D_{oy}^\alpha$  is Caputo differentiating operator fractional order  $\alpha$ , ( $0 < \alpha < 1$ ).

We enter designations  $\Omega^+ = \Omega \cap \{y > 0\}$ ,  $\Omega^- = \Omega \cap \{y < 0\}$  ?  $I = \{(x, y) : y = 0, 0 < x < l\}$ .

We assume that  $f(x)$  is given sufficiently smooth function.

**Direct problem** To find a function  $u(x, y)$  for equation (1), with the following properties:

1.  $f(x) \in C[0, l]$ ,  $g_1(y) \in C^1[0, h]$ ,  $g_2(y) \in C[-\frac{l}{2}, 0]$ ;
2.  $u \in C(\bar{\Omega}^+)$ ,  $u_{xx} \in C(\Omega^+)$ ,  ${}_C D_{oy}^\alpha u \in C(\Omega^+ \cup I)$ ,  $u \in C(\bar{\Omega}^-) \cap C^2(\Omega^-)$ ;
3.  $u(x, y)$  satisfy boundary conditions

$$u(0, y) = \phi_1(y), \quad u(l, y) = \phi_2(y), \quad 0 \leq y \leq h;$$

$$u(x, -x) = \psi(x), \quad 0 \leq x \leq l;$$

and the following gluing conditions

$$u(x, +0) = \delta_1 u(x, -0) + \delta_2, \quad 0 \leq x \leq l;$$

$$\lim_{y \rightarrow +0} y^{1-\alpha} u_y(x, y) = \lambda u_y(x, -0) + r(x, u(x, -0)), \quad 0 < x < l,$$

where  $\lambda$ ,  $\delta_1$ ,  $\delta_2 = \text{const}$ ,  $r(x, z(x))$ ,  $p_i(x, y, z(x))$ ,  $f(x)$ ,  $\psi(x)$ ,  $\phi_i(y)$ ,  $g_i(y)$  ( $i = 1, 2$ ), are given sufficiently smooth function, besides  $\delta_1 > 1, \lambda \neq 0$ .

Now we assume that  $f(x)$  and  $u(x, y)$  are unknown function.

**Inverse problem** Required to find a pair of functions  $\{f(x), u(x, y)\}$  for equation (1), satisfy all conditions of the Direct problem and additional condition

$$u_n(x, y) \Big|_{B_1C} = \psi_1(x), \quad 0 \leq x \leq \frac{l}{2}, \quad u_n(x, y) \Big|_{A_1C} = \psi_2(x), \quad \frac{l}{2} \leq x \leq l;$$

where  $\psi_i(x)$  ( $i = 1, 2$ ), are given functions, satisfying matching condition  $\psi_1\left(\frac{1}{2}\right) = \psi_2\left(\frac{1}{2}\right)$ .

REMARK. Under certain conditions on the given functions and on the boundary of the domain, the local unique solvability of the formulated problem is proved. Unique solvability is proved by the method of integral equations using the principle of compressed mappings.

**Keywords:** diffusion-wave equation, direct and inverse problems, nonlinear loaded terms, fractional differentiation, local problem, nonlinear gluing condition, unique solvability, method of successive approximations

**2020 Mathematics Subject Classification:** 35Q79, 35K05, 35K20

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## Boundedness and Compactness of Commutator Bilinear Riesz Potential in Generalized Morrey Spaces

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In this paper, we obtain that  $b \in VMO(\mathbb{R}^n)$  the commutator  $[b; I_\alpha]$  is compactness in generalized Morrey space from  $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$  to  $M_q^{w, \cdot}$ , for some appropriate indices  $p_1; p_2; q; w; w_1; w_2; \alpha$ .

The main goal of this paper is to find sufficient conditions for the commutator of bilinear  $[b, I_\alpha]$  to be precompact in the generalized Morrey space  $M_p^w(\mathbb{R}^n)$ .

Let  $1 \leq p \leq \infty$ ,  $w$  be a measurable non-negative function on  $(0, \infty)$  that is not equivalent to zero. The generalized Morrey space  $M_p^{w(\cdot)} \equiv M_p^{w(\cdot)}(\mathbb{R}^n)$  is defined as the set of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite norm  $\|f\|_{M_p^{w(\cdot)}} \equiv \sup_{x \in \mathbb{R}^n} \sup_{r > 0} \left( w(r) \|f\|_{L_p(B(x,r))} \right) < \infty$

For  $0 < \alpha < 2n$ , the bilinear fractional integral is defined by  $I_\alpha(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}} dydz$

For a function  $b \in L_{loc}(\mathbb{R}^n)$  by  $M_b$  denote multiplier operator  $M_b f = bf$ , where  $f$  is measurable function. Then the commutator between  $I_\alpha$  and  $M_b$  is defined by  $[b, I_\alpha]_1(f, g)(x) = M_b I_\alpha - I_\alpha M_b = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y)-b(x))f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}} dydz$ ,  
 $[b, I_\alpha]_2(f, g)(x) = M_b I_\alpha - I_\alpha M_b = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(z)-b(x))f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}} dydz$

It is said that the function  $b(x) \in L_\infty(\mathbb{R}^n)$  belongs to the space  $BMO(\mathbb{R}^n)$ , if  $\|b\|_* = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = \sup_{Q \in \mathbb{R}^n} M(b, Q) < \infty$ , where  $Q$  - cube  $\mathbb{R}^n$  and  $b_Q = \frac{1}{|Q|} \int_{\mathbb{R}^n} f(y)dy$ .

By  $VMO(\mathbb{R}^n)$  we denote the  $BMO$ -closure  $C_0^\infty(\mathbb{R}^n)$ , where  $C_0^\infty(\mathbb{R}^n)$  the set of all functions from  $C^\infty(\mathbb{R}^n)$  with compact support. Through the  $\chi(A)$  denotes the characteristic function of the set  $B \subset \mathbb{R}^n$ , and  ${}^c A$  denotes the complement of  $A$ .

The main goal of this paper is to find sufficient conditions for the commutator of bilinear  $[b, I_\alpha]$  to be precompact in the generalized Morrey space  $M_p^w(\mathbb{R}^n)$ .

Let  $1 \leq p \leq \infty$ ,  $w$  be a measurable non-negative function on  $(0, \infty)$  that is not equivalent to zero. The generalized Morrey space  $M_p^{w(\cdot)} \equiv M_p^{w(\cdot)}(\mathbb{R}^n)$  is defined as the set of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite norm

$$\|f\|_{M_p^{w(\cdot)}} \equiv \sup_{x \in \mathbb{R}^n} \sup_{r > 0} \left( w(r) \|f\|_{L_p(B(x,r))} \right) < \infty$$

**Theorem 1.** [1] Let  $1 \leq p \leq \infty$  and  $w \in \Omega_{p,\infty}$ . Let's assume that the set  $S \subset M_p^{w(\cdot)}$  satisfies the following conditions:

$$\begin{aligned} \sup_{f \in S} \|f\|_{M_p^{w(\cdot)}} &< \infty, \\ \lim_{|u| \rightarrow 0} \sup_{f \in S} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}} &= 0, \\ \lim_{r \rightarrow \infty} \sup_{f \in S} \|f \chi_{c_{B(0,r)}}\|_{M_p^{w(\cdot)}} &= 0. \end{aligned}$$

Then  $S$  is a pre compact set in  $M_p^{w(\cdot)}$ .

Suppose that the continuous increasing functions  $w_1, w_2$  in  $[0; \infty)$  satisfy the following conditions ( $j = 1, 2$ ):

- a)  $w_j(0) = 0$ ;
- b)  $\lim_{r \rightarrow \infty} w_j(r) < \infty$ ;
- c) There exists a constant  $D$ , satisfying  $1 \leq D < 2^n$ , such that  $w_j(2r) \leq Dw_j(r)$  for any  $r > 0$ ;
- d)  $w(r)^{\frac{1}{p}} = w_1(r)^{\frac{1}{p_1}} w_2(r)^{\frac{1}{p_2}}$

**Theorem 2.** Let  $0 < \alpha < 2n, 0 < \lambda < n, \frac{1}{2} < p < \frac{n-\lambda}{\alpha}, 1 < p_1, p_2 < \infty, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, 1 < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ , functions  $w, w_1, w_2 \in \Omega_{p,\infty}$  satisfying conditions (a)-(d) for any  $r > 0$ . Then  $I_\alpha$  is bounded bilinear operator from  $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$  to  $M_q^w$ . For  $b \in BMO(\mathbb{R}^n)$ , the commutator  $[b, I_\alpha]_i$  is also bounded from  $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$  to  $M_q^w, i = 1, 2$ ,

$$(1) \quad \|[b, I_\alpha]_i(f, g)\|_{M_q^w} \leq C \|b\|_{BMO} \|f\|_{M_{p_1}^{w_1}} \|g\|_{M_{p_2}^{w_2}}$$

**Theorem 3.** Let  $0 < \alpha < 2n, 0 < \lambda < n, \frac{1}{2} < p < \frac{n-\lambda}{\alpha}, 1 < p_1, p_2 < \infty, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, 1 < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ , function  $w, w_1, w_2 \in \Omega_{p,\infty}$  satisfying conditions (a)-(d) for any  $r > 0$  and  $b \in VMO(\mathbb{R}^n)$ , (1). Then commutator  $[b, I_\alpha]_i$  is a compact from  $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$  to  $M_q^w, i = 1, 2$ .

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**Keywords:** commutator, single integral operator, compactness, global Morrey-type space.

**2020 Mathematics Subject Classification:** 35Q79, 35K05, 35K20

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# Solution of the Cauchy Problem of the Dynamics of a Thermoelastic Rod Using the Vladimirov Method

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Nonstationary boundary value problems of the dynamics of thermoelastic rods are considered. Based on the generalized function method, a solution to the Cauchy problem was constructed using a model of uncoupled thermoelasticity under the influence of non-stationary force and thermal loads of various types. Regular integral representations of the solution to the Cauchy problem for a thermoelastic rod are obtained.

**1. Statement of the Cauchy problem for the equations of uncoupled thermoelasticity.** We consider a thermoelastic rod, the equations of state of which have the form [1]:

$$(2) \quad \begin{cases} \rho c^2 u_{,xx} - \rho u_{,tt} - \gamma \theta_{,x} + \rho F_1(x, t) = 0 \\ \theta_{,xx} - k^{-1} \theta_{,t} + F_2(x, t) = 0 \end{cases}$$

where  $x \in R^1$ ,  $t \geq 0$ .  $F_1(x, t)$  -longitudinal component of external force per unit length,  $F_2(x, t)$  - quantity characterizing the power of the heat source.

Thermoelastic stresses in the rod are related to displacement by the Duhamel-Neumann law:

$$(3) \quad \sigma(x, t) = \rho c^2 u_{,x} - \gamma \theta$$

The initial conditions are known:

$$(4) \quad \begin{aligned} u(x, 0) = u_0(x), u_{,t}(x, 0) = \nu_0(x), \theta(x, 0) = \theta_0(x), \\ u_0(x) \in C^2(R^1), \theta_0(x) \in C^2(R^1), \nu_0(x) \in C^2(R^1). \end{aligned}$$

$C^n(R^1)$  where is the space of functions differentiable up to the nth order on  $R^1$ .

It is required to find solutions to equations (1) with initial conditions (3), which satisfy the radiation conditions: under  $u(x, t) \rightarrow 0$ ,  $\theta(x, t) \rightarrow 0$ , under  $|x| \rightarrow \infty$ ,  $\forall t$  the action of arbitrary forces and heat sources:  $F_j(x, t) = L_1(R^1)$ , under  $j = 1, 2$ .

**2. Statement of the Cauchy problem in the space of generalized functions.** To solve the problem we use the method developed by V.S. Vladimirov. to solve the Cauchy problem of wave equations [2]. Let us introduce the following regular generalized functions:

$$(5) \quad \hat{u}(x, t) = u(x, t)H(t), \hat{\theta}(x, t) = \theta(x, t)H(t), \hat{F}(x, t) = F(x, t)H(t)$$

System (1) in the space of generalized functions will have the following form:

$$(6) \quad \begin{aligned} c^2 \hat{u}_{,xx} - \hat{u}_{,tt} - \tilde{\gamma} \hat{\theta}_{,x} &= -F_1(x, t)H(t) - \nu_0(x)\delta(t) - u_0(x)\delta'(t) = -\hat{F}_1(x, t) \\ \frac{\partial \hat{\theta}}{\partial x^2} - k^{-1} \frac{\partial \hat{\theta}}{\partial t} &= -F_2(x, t)H(t) + k^{-1} \theta_0(x)\delta(t) = -\hat{F}_2(x, t). \end{aligned}$$

where  $\tilde{\gamma} = \frac{\gamma}{\rho}$ . The right-hand sides of these equations (5) include the initial conditions as singular mass forces and heat sources:

$$(7) \quad \begin{aligned} \hat{F}_1(x, t) &= F_1(x, t)H(t) + \nu_0(x)\delta(t) + u_0(x)\delta'(t), \\ \hat{F}_2(x, t) &= F_2(x, t)H(t) + k^{-1}\theta_0(x)\delta(t). \end{aligned}$$

The solution to this problem in the space of generalized functions has the form of tensor-functional convolution [1]:

$$(8) \quad \begin{aligned} \hat{u}(x, t) &= u(x, t)H(t) = \hat{U}_1^k(x, t) * \hat{F}_{\Delta k}(x, t), (k = 1, 2), \\ \hat{\theta}(x, t) &= \theta(x, t)H(t) = \hat{U}_2^k(x, t) * \hat{F}_{\Delta k}(x, t), (k = 1, 2). \end{aligned}$$

**3. Green’s tensor of the equations of unbound thermoelasticity.** Green’s tensor  $U_i^j(x, t)$  is a matrix of fundamental solutions of system (1) under the action of pulsed concentrated forces and a heat source of the form:

$$(9) \quad F_1 = \delta(x)\delta(t)\delta_1^j, F_2 = \delta(x)\delta(t)\delta_2^j, j = 1, 2$$

The Green’s tensor is constructed and has the following form [3]:

$$(10) \quad \begin{aligned} U_1^j(x, t) &= \delta_1^j k^{-1} \frac{\partial \Sigma_1}{\partial t} - \delta_1^j \frac{\partial \Sigma_1}{\partial x^2} - \delta_2^j \tilde{\gamma} \frac{\partial \Sigma_1}{\partial x} - \delta_1^j \Sigma_3(t)\delta(x) + \delta_2^j \tilde{\gamma} \frac{\partial \Sigma_2}{\partial x}, \\ U_2^1 &= 0, U_2^2 = c^2 \Sigma_3(x, t) + c^2 \frac{\partial^2 \Sigma_2}{\partial x^2} - \frac{\partial^2 \Sigma_2}{\partial t^2}, \dots j = 1, 2. \end{aligned}$$

Knowing the Green’s tensor, it is possible to construct a solution to system (6) for any sources in the form of a tensor-functional convolution (7).

**4. Solutions to the Cauchy problem.** To obtain an integral representation of the generalized solution, we take convolutions (7) taking into account (6), using the properties of the  $\delta$ - function and its derivative:

$$(11) \quad \begin{aligned} \hat{u}(x, t) &= u(x, t)H(t) = \hat{U}_1^1(x, t) * F_1(x, t)H(t) + \hat{U}_1^1(x, t) *_{x} \nu_0(x) + \\ &\frac{\partial}{\partial t} \hat{U}_1^1(x, t) *_{x} u_0(x) + \hat{U}_1^2(x, t) *_{x} F_2(x)H(t) + \hat{U}_1^2(x, t) *_{x} k^{-1} \theta_0(x) \end{aligned}$$

$$(12) \quad \hat{\theta}(x, t) = \theta(x, t)H(t) = \hat{U}_2^2(x, t) * F_2(x, t)H(t) + \hat{U}_2^2(x, t) *_{x} k^{-1} \theta_0(x)$$

I would like to note that all the initial conditions of the Cauchy Problem are included in the right side of relations (10), (11).

So, using the apparatus of generalized function theories, we solved the Cauchy problem. The solutions obtained make it possible to find the stress-strain state in the rod under the action of various common power and heat sources using formulas and theorems, i.e. if we pose the Cauchy Problem under new initial conditions (3). We substitute these data into the right side of the resulting formulas (10) and (11) and obtain new solutions to the Cauchy Problem.

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**Keywords:** Cauchy problem, rod, uncoupled thermoelasticity, generalized function method, heat flow, temperature, displacement and stress.

**2020 Mathematics Subject Classification:** 74A15, 65R20, 35L05, 35K05.

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# Approximation of Functions of Two Variables of Bounded P-Fluctuation by Polynomials with Respect to Walsh Systems

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IN THIS PAPER, WE PROVE DIRECT AND CONVERSE THEOREMS OF APPROXIMATION OF FUNCTIONS OF TWO VARIABLES OF BOUNDED P-FLUCTUATIONS BY WALSH POLYNOMIALS IN THE NORM OF THE SPACE.

Let  $1 \leq p < \infty$  and the function  $f(x, y)$  be defined on the set  $[0, 1]^2$  and for any set  $I$  denote by  $I_{j_1, j_2}^{(n_1, n_2)}$  rectangular  $\left[ \frac{j_1-1}{2^{n_1}}, \frac{j_1}{2^{n_1}} \right) \times \left[ \frac{j_2-1}{2^{n_2}}, \frac{j_2}{2^{n_2}} \right)$ . Denote by  $osc(f, [a, b]^2) = \sup_{(x,y), (x',y') \in I} |f(x, y) - f(x', y')|$ .

By definition,  $\kappa_p(f, n_1, n_2) := \left( \sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left( osc(f, I_{j_1, j_2}^{(n_1, n_2)}) \right)^p \right)^{1/p}$ . If  $V_p(f) := \sup_{\substack{n_1 \in \mathbb{P} \\ n_2 \in \mathbb{P}}} \kappa_p(f, n_1, n_2) < \infty$ , then  $f(x, y)$  is called the function of bounded p-fluctuation. In one variable case, the definition was introduced by Onneweer and Waterman [1] Now we introduce a discrete modulus of continuity  $V_p(f)_{n_1, n_2} = \sup_{\substack{k_1 \geq n_1 \\ k_2 \geq n_1}} \kappa_p(f, k_1, k_2)$ . The set of functions  $f(x, y)$ , for which  $V_p(f) < \infty$ , we denote by  $FV_p[0, 1]^2$  ( $1 \leq p < \infty$ ). Let us introduce one more discrete group modulus of continuity related to the space  $FC_p[0, 1]^2$  ( $1 < p < \infty$ ), by the formula

$$V_p(f)_{n_1, n_2}^* = \sup_{\substack{0 \leq h_1 < \frac{1}{2^{n_1}} \\ 0 \leq h_2 < \frac{1}{2^{n_2}}}} \|f(x_1 \oplus h_1, x_2 \oplus h_2) - f(x_1, x_2)\|_{p, F}.$$

**Theorem 1.** *Let  $1 < p < \infty$ ,  $n, m \in \mathbb{N}$ ,  $f \in FV_p[0, 1]^2$ . Then the following inequality is valid:  $\frac{1}{2} V_p(f)_{m, n}^* \leq E_{2^m, 2^n}(f)_{p, F} \leq V_p(f)_{m, n}^*$ .*

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**Keywords:** functions of bounded p-fluctuation, Walsh systems, discrete modulus of continuity.

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# Nijenhuis Geometry and Geodesically Equivalent Metrics

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Let  $M^n$  be a smooth manifold and  $L$  an operator filed on it.

DEFINITION. Nijenhuis torsion  $N_L$  is a tensor type of  $(1, 2)$  which in the invariant form can be defined by following formula

$$N_L = L^2[u, v] + [Lu, Lv] - L[Lu, v] - L[u, Lv],$$

where  $u$  and  $v$  are a smooth vector fields.

DEFINITION. Operator field  $L$  is Nijenhuis operator if  $N_L \equiv 0$ .

DEFINITION. Two metrics  $g$  and  $\bar{g}$  are called Geodesically Equivalent if they share same geodesics viewed as unparameterized curves.

Let  $g$  and  $\bar{g}$  are metrics on the smooth manifold  $M$ . Then the condition of the Geodesic Equivalence can be written by the following equation:

$$\nabla_k^g R_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where  $R = |\frac{\det \bar{g}}{\det g}|^{\frac{1}{n+1}} g \bar{g}^{-1} g$ ,  $\lambda_i = \frac{\partial R_{sj} g^{sj}}{\partial x^i}$ .

In the coordinates it's a system of linear PDE for components of the metric  $g$ .

$$g_{is} \frac{\partial L_j^s}{\partial x^k} + g_{js} \frac{\partial L_i^s}{\partial x^k} - \left( \frac{\partial g_{sk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^s} \right) L_j^s - \left( \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right) L_i^s = \frac{\partial \text{tr} L}{\partial x^i} g_{jk} + \frac{\partial \text{tr} L}{\partial x^j} g_{ik}$$

**Theorem 1.** [1]  $L = g^{-1}R$  is Nijenhuis operator.

DEFINITION. A metric  $g$  and a Nijenhuis operator  $L$  are said to be *geodesically compatible*, if  $L$  is  $g$ -self-adjoint and the metric  $\bar{g} = \frac{1}{\det L} g L^{-1}$  is geodesically equivalent to  $g$ .

Let  $\chi(t) = \det(tE - L) = t^n + \sigma_1 t^{n-1} + \dots + \sigma_n$  be characteristic polynomial of the Nijenhuis operator  $L$ .

**Theorem 2.** For geodesically compatible pair  $(L, g)$  true next formula:

$$\det g = f_1(\lambda_1) \dots f_n(\lambda_n) (\det d\Phi)^2,$$

where  $\lambda_i$  are eigenvalues and  $\Phi : M^n \rightarrow \mathbb{R}^n$ ,  $\Phi = (\sigma_1, \dots, \sigma_n)$ ,  $f_i$  are arbitrary almost everywhere smooth functions.

**Theorem 3.** Let  $L$  be a Nijenhuis operator, then:

1. If  $L$  is algebraically generic, then there exists geodesically compatible metric  $g$  and geodesic flow of the  $g$  is integrable;
2. If  $L$  is gl-regular ([1]), then also we can find geodesically compatible metric  $g$  and geodesic flow of the  $g$  is integrable;

In the talk I will show examples of the Nijenhuis operators when there does not exist geodesically compatible metric.

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**Keywords:** Integrable systems, Nijenhuis geometry, Nijenhuis operators, Geodesically Equivalent Metrics, Geodesic flow.

**2020 Mathematics Subject Classification:** 53B99, 53A45, 37K10

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## On the Fourier Multipliers with Respect to the General Haar System

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This work is devoted to the study of the Fourier series multipliers in the generalized Haar system in Lebesgue and Lorentz spaces.

Let us consider the sequence  $\lambda = \{\lambda_{l,h}^k\}_{k=1, l=0, h=1}^{\infty, m_k-1, p_{k+1}-1}$ . Any sequence  $\lambda$  generates an operator  $\Lambda$ , called a multiplier, which is defined on polynomials in the generalized Haar system by the following way

$$\Lambda \left( \sum_{k=1}^N \sum_{l=0}^{m_k-1} \sum_{h=1}^{p_{k+1}-1} a_{l,h}^k(f) \chi_{l,h}^k(x) \right) = \sum_{k=1}^N \sum_{l=0}^{m_k-1} \sum_{h=1}^{p_{k+1}-1} \lambda_{l,h}^k a_{l,h}^k \chi_{l,h}^k(x),$$

here  $\chi_n(x) := \chi_{l,h}^k(x)$  is the generalized Haar system, see e.g. [1] and [4].

Let us consider the spaces  $n_{p,r}(L_q)$ , which are the essence of the proposed method. These spaces are a version of the net spaces, which was introduced and studied in the papers [2] and [3]. Net spaces are an important research tool in the theory of Fourier series, operator theory, and other areas.

Let  $1 < p < q < \infty, \frac{1}{p'} = 1 - \frac{1}{p}, 0 < r \leq \infty$ . We say that function  $f$  belongs to the net space  $n_{p,r}(L_q)$  if  $f \in L_1$  and

$$\|f\|_{n_{p,r}(L_q)} := \left( \sum_{k=1}^{\infty} \left( m_k^{\frac{1}{q} + \frac{1}{p'}} \sup_{n \geq k} \frac{1}{m_n} \|S_n(f)\|_{L_q} \right)^r \right)^{\frac{1}{r}} < \infty$$

for  $0 < r < \infty$ ,

$$\|f\|_{n_{p,r}(L_q)} := \sup_k m_k^{\frac{1}{q} - \frac{1}{p}} \|S_k(f)\|_{L_q} < \infty$$

for  $r = \infty$ .

Moreover, for the proof of the main theorem, we need some embedding and interpolation results of independent interest.

**Theorem 1.** Let  $1 < p < q < \infty, 0 < \tau \leq \infty, \{p_k\}$  be a sequence of natural numbers such that  $p_0 = 1, 2 \leq p_k \leq c < +\infty, m_0 = 1, m_k = m_{k-1}p_k, k \in \mathbb{N}$ . If  $f \in L_{p,\tau}[0, 1]$ , then

$$L_{p,\tau} \hookrightarrow n_{p,\tau}(L_q),$$

here the constant  $c$  depends only on the parameters  $p, q$  and  $\tau$ .

**Coorollary 1.** Let  $1 < p < q < \infty, 0 < \tau \leq \infty, \{p_k\}$  be a sequence of natural numbers such that  $p_0 = 1, 2 \leq p_k \leq c < +\infty, m_0 = 1, m_k = m_{k-1}p_k, k \in \mathbb{N}$ . If  $f \in L_{p,\tau}[0, 1]$ , then

$$\left( \sum_{k=0}^{\infty} \left( m_k^{\left(\frac{1}{q} - \frac{1}{p}\right)} \|S_k(f)\|_{L_q} \right)^{\tau} \right)^{\frac{1}{\tau}} \leq c \|f\|_{L_{p,\tau}},$$

here the constant  $c$  depends only on the parameters  $p, q$  and  $\tau$ .

A new theorem on sufficient conditions for the sequence to belong to the class  $m(L_{p,r} \rightarrow L_{q,s})$  of multipliers of Fourier series concerning the generalized Haar system is proved.

**Theorem 2.** Let  $1 < p < \infty, 0 < r, s \leq \infty, \frac{1}{\tau} = \left(\frac{1}{s} - \frac{1}{r}\right)_+ = \max\left\{\frac{1}{s} - \frac{1}{r}, 0\right\}$  and  $\{p_k\}$  be a sequence of natural numbers such that  $p_0 = 1, 2 \leq p_k \leq c < +\infty, m_0 = 1$ ,

$m_k = m_{k-1}p_k$ ,  $k \in \mathbb{N}$ . Let the sequence of complex numbers  $\lambda = \{\lambda_k\}_{k \in \mathbb{N}}$  satisfy the following condition

$$\left( \sum_{k=0}^{\infty} \left( m_k^{\left(\frac{1}{p} - \frac{1}{q}\right)} \sup_{m_k+1 \leq j \leq m_{k+1}} |\lambda_k^j| \right)^\tau \right)^{\frac{1}{\tau}} \leq A, \text{ for } 0 < \tau < \infty$$
$$\sup_{\substack{0 \leq k \leq \infty \\ m_k+1 \leq j \leq m_{k+1}}} m_k^{\left(\frac{1}{p} - \frac{1}{q}\right)} |\lambda_k^j| \leq A, \text{ for } \tau = +\infty.$$

Then  $\lambda \in m(L_{p,r} \rightarrow L_{q,s})$  and

$$\|\lambda\|_{m(L_{p,r} \rightarrow L_{q,s})} \leq cA.$$

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**Keywords:** Fourier coefficients, generalized Haar system, Fourier multiplier, Lorentz spaces.

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# Radial and Logarithmic Refinements of the Weighted Hardy Inequality

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In this paper, we obtain weighted versions of radial and logarithmic refinements of the following Hardy’s inequality using the factorization method of differential operators from [1]-[3]:

$$\int_{\Omega} |(\nabla f)(x)|^2 d^n x \geq \int_{\Omega} |x - x_0|^{-2} |f(x)|^2 \left\{ \frac{(n-2)^2}{4} + \frac{1}{4} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma/|x-x_0|)]^{-2} \right\} d^n x, \tag{1}$$

valid for  $f \in C_0^\infty(\Omega)$ , assuming that  $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}, n \geq 2$ , is open and bounded with  $x_0 \in \Omega, m \in \mathbb{N}$ , and the logarithmic terms  $\ln_k(\gamma/|x-x_0|), k \in \mathbb{N}$ .

Moreover, we discuss generalizations of these results on homogeneous Lie groups.

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**Keywords:** factorization method, Hardy’s inequality, homogeneous Lie group, stratified group.

**2010 Mathematics Subject Classification:** 22E30,26D10

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# Large Time Asymptotes to the Cauchy Problem for Doubly Nonlinear Parabolic Equation with Variable Density and Absorption

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The following Cauchy problem

$$\rho_2(x) \partial_t u = u^q \operatorname{div} \left( \rho_1(x) u^{m-1} \left| \nabla u^k \right|^{p-2} \nabla u \right) - \rho_3(x) t^l u^\beta, u(t_0, x) = u_0(x), x \in R^N \quad (1)$$

is considered in  $Q = \{(t, x) : t > 0, x \in R^N\}$ ,

where  $m, k \geq 1, p \geq 2, n, l_i, l, \beta$  ( $i = 1, 2$ ) are the given numerical parameters

$$\rho_1(x) = |x|^n, \rho_{i+1}(x) = |x|^{-l_i}, i = 1, 2, q \in (0, 1).$$

The different qualitative properties of the solution problem (1) in one and many dimensional cases intensively studied by many authors [see [1-3] and references therein]. In this work estimates of solutions and a free boundary, behaviors of solutions of the problem (1) for slowly ( $k(p-2) + m - 1 > 0$ ), fast ( $k(p-2) + m - 1 < 0$ ), a critical diffusion ( $k(p-2) + m - 1 = 2$ ) cases, depending on value of density  $p > n + l, p = n + l$  (singular) established. The problem choosing of an initial approximation for iteration process solved. The results of numerical experiments discussed.

**Keywords:** global solvability, estimate solution, critical, singular cases, asymptotic, numerical analysis..

**2020 Mathematics Subject Classification:** 35C06, 35D30, 35K55

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# On the Nonlocal Boundary Value Problem with Samarskii–Ionkin Condition I for the Elliptic Differential Equations

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Various nonlocal boundary value problems with Samatskii-Ionkin condition for partial differential have been investigated by many researchers (see, e.g. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], and the references given therein).

In the present paper, the nonlocal boundary value problem with Samarskii–Ionkin condition I for elliptic equations

$$(13) \quad \begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = f(t), 0 < t < T, \\ u(0) = u(T) + \varphi, u'(T) + \mu u(T) = \psi \end{cases}$$

in a Banach space  $E$  with stronly positive operator  $A$  and  $\mu \geq 0$  was investigated.

A function  $u(t)$  is called a solution of the problem (13) if the following conditions are satisfied:

i)  $u(t)$  is twice continuously differentiable on the interval  $(0,T)$  and continuously differentiable on the segment  $[0, T]$ . The derivatives at the endpoints of the segment are understood as the corresponding unilateral derivatives.

ii) The element  $u(t)$  belongs to  $D(A)$  for all  $t \in [0, T]$ , and the function  $Au(t)$  is continuous on the segment  $[0, T]$ .

iii)  $u(t)$  satisfies the equations and the nonlocal boundary condition (13).

In the present paper, we study the well-posedness of the nonlocal boundary value problem (13). The main theorem on well-posedness of problem (13) is established. In practice, the coercive stability estimates for solution of four types of nonlocal boundary value problems with Samatskii-Ionkin condition I for elliptic differential equations are proved.

**Keywords:** elliptic equations, coercive stability, well-posedness, stronly positive operator

**MSC[2010]:** 35L10, 35L90, 35B35

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# The Absolute Stable Difference Schemes to the Time-Dependent Source Identification Problem for the Schrödinger Equation

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In this study, we consider the time-dependent source identification problem

$$\begin{cases} i \frac{du(t)}{dt} + Au(t) = p(t)q + f(t), & t \in (0, T), \\ u(0) = \varphi, B[u(t)] = \psi(t), & t \in [0, T] \end{cases}$$

for Schrödinger differential equation in a Hilbert space  $H$  with the self-adjoint positive definite operator  $A$  with dense domain  $D(A)$  in  $H$ . Here  $B : H \rightarrow \mathbb{R}$  is a given linear bounded functional and  $\psi(t) : [0, T] \rightarrow \mathbb{R}$  is a given smooth function and  $q \in D(A)$ ,  $Bq \neq 0$ .

Single-step absolute stable difference schemes for the approximate solutions to the time-dependent source identification problem(1) are investigated. The first and second order of accuracy implicit and  $r$ -modified Crank-Nicolson difference schemes are presented. The stability of these difference schemes are established. In applications, theorems on the stability inequalities for the numerical solutions of several types of source identification problems for Schrödinger equations are proved.

**Keywords:** Schrödinger equation, source identification problem, Hilbert space, positive operator, absolute stable, difference schemes.

**2020 Mathematics Subject Classification:** 35A22, 35B30, 35F10

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## On the Asymptotic Solutions of Perturbation Problems for Hyperbolic Equations

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The abstract initial value problem

$$\begin{cases} \varepsilon^2 u''(t) + Au(t) = f(t), 0 < t < T, \\ u(0) = \varphi, u'(0) = \psi \end{cases}$$

for hyperbolic equations is considered. The study is done in a Hilbert space  $H$  with the self adjoint positive definite operator  $A$ . A small parameter  $\varepsilon \in (0, \infty)$  is used in the equation. The solution of the problem using an asymptotic formula with a small parameter is introduced. The study presents the general perturbation theory of uniform difference schemes on hyperbolic PDEs. Convergence estimates are obtained and some numerical verifications which verify the theoretical results are presented.

**Keywords:** hyperbolic equations, Cauchy problem, asymptotic formula, uniform difference schemes.

**2020 Mathematics Subject Classification:** 47A55, 39A30, 34K25

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## Source Identification Problem for Multidimensional Reverse Parabolic Equation

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In the paper [1] source identification problem for parabolic equation with multipoint nonlocal condition was studied. The paper [2] is devoted to the well-posedness of direct nonlocal problems for a reverse parabolic equation with integral type nonlocal condition. The aim of the present work is to study the source identification problem for the multidimensional reverse parabolic equation.

We consider in  $[0, 1] \times \bar{\Omega}$ , the source identification problem to find unknown functions  $v$  and  $p$  for the multi dimensional reverse parabolic equation with the following multi point nonlocal and Dirichlet boundary conditions

$$(14) \quad \left\{ \begin{array}{l} \frac{\partial v(t,x)}{\partial t} + \sum_{i=1}^n (a_r(x)v_{x_i}(t,x))_{x_i} - \sigma v(t,x) = f(t,x) + p(x), \\ x = (x_1, \dots, x_n) \in \Omega, \quad 0 < t < 1, \\ v(1,x) = \sum_{k=1}^r \mu_k v(s_k, x) + \psi, \quad v(0,x) = \varphi(x), \quad x \in \bar{\Omega}, \\ v(t,x) = 0, \quad x \in S. \end{array} \right.$$

Here  $\Omega = (0, 1)^n$  is the unit open cube in the  $\mathbb{R}^n$  with boundary  $S = \partial\Omega, \bar{\Omega} = \Omega \cup S$  and  $\gamma_1, \gamma_2, \dots, \gamma_r, \mu_1, \mu_2, \dots, \mu_r$  are given numbers and inequalities

$$(15) \quad \sum_{k=1}^r |\mu_k| < 1, 0 \leq \gamma_1 < \gamma_2 < \dots < \gamma_r < 1.$$

Let  $a_r(x) \geq a_0 > 0, r = 1, \dots, n$  be given functions,  $\sigma$  is a known positive number.

**Theorem.** Assume that the inequalities (15) are valid,  $\varphi \in L_2(\Omega), \psi \in W_2^2(\Omega), f \in C^\alpha(L_2(\Omega))$  are given. Then, the source identification problem (14) is uniquely solvable and for its solution the following stability estimates are valid:

$$(16) \quad \|p\|_{L_2(\Omega)} \leq M \left[ \|\varphi\|_{L_2(\Omega)} + \|\psi\|_{W_2^2(\Omega)} + \frac{1}{\alpha} \|f\|_{C^\alpha(L_2(\Omega))} \right]$$

$$(17) \quad \|v\|_{C(L_2(\Omega))} \leq M \left[ \|\varphi\|_{L_2(\Omega)} + \|\psi\|_{L_2(\Omega)} + \|f\|_{C(L_2(\Omega))} \right],$$

where positive number  $M$  is independent of  $f(t, x), \psi(x), \varphi(x)$ , and  $\alpha$ .

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**Keywords:** reverse parabolic equation, source identification problem, stability, inverse problem, multi point conditions, .

**2020 Mathematics Subject Classification:** 35M12

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## Third-order weighted essentially nonoscillatory schemes for advection-diffusion equation

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Advection-diffusion equations are widely used in the modeling of many physical phenomena, such as in fluid dynamics, semiconductor physics, thermal heat diffusion, flow through porous media, the spread of pollution in rivers, etc. Since the analytical solutions of advection-diffusion equations are not easy to find, one needs to solve them by using numerical methods.

Weighted essentially non-oscillatory (WENO) schemes [1] have become popular techniques for the numerical solutions of hyperbolic conservation laws. It is known that the standard WENO schemes are unstable for advection-diffusion problems. This paper shows the extension of the third-order finite volume WENO scheme [2] to solve one-dimensional advection-diffusion equations.

**Keywords:** Advection-diffusion equation, WENO3 scheme

**2020 Mathematics Subject Classification:** 65M08, 65M20, 65D05

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# Investigation of Discrete Analogs of Integral Geometry Problem with a Weight Function

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The interest in reconstruction problems has grown tremendously in the last four decades, stimulated by the spectrum of new modalities of image reconstruction. These are X-ray, MRI, gamma and positron radiography, ultrasound, seismic tomography, electron microscopy, synthetic radar imaging and others. The physical principles of these methods are very different; however their mathematical models and solution methods have very much in common. The umbrella name reconstructive integral geometry is used to specify the variety of these problems and methods [1]-[3].

With fairly general assumptions about the family of curves and the weight function, the problem of integral geometry is reduced to a boundary value problem for a second-order partial differential equation. Estimates of the conditional stability of discrete analogs of a two-dimensional integral geometry problem with a weight function on the space of sufficiently smooth functions are obtained.

The results of studying the problem under consideration contribute to the development of the theory and practical implementation of integral geometry problems in various fields of science and technology.

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**Keywords:** integral geometry, stability estimate, difference problem, finite-difference analogue, differential-difference analogue.

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## On Some Nonlocal Boundary Value Problems for Second Order Elliptic Systems

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We consider an elliptical system

$$a_{11} \frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21}) \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

with constant coefficients  $a_{ij} \in \mathbb{R}^{l \times l}$ .

Its solution is understood as  $l$ -vector function  $u = (u_1, \dots, u_l)$  from class  $C^2$  in the area under consideration, satisfying identical to equation (1).

The ellipticity condition is that  $\det a_{22} \neq 0$  and the characteristic polynomial

$$\chi(z) = \det P(z), \quad P(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2, \tag{2}$$

of degree  $2l$  have no real roots.

Let  $\nu_1, \dots, \nu_m$  be all distinct roots of the polynomial  $\chi$ , lying in the upper half-plane  $\text{Im } z > 0$ , which we also call the eigenvalues of the matrix polynomial  $P(z)$ .

**Lemma 1.** *There are matrices  $b, J \in \mathbb{C}^{l \times l}$ , such that*

$$\det \begin{pmatrix} b & \bar{b} \\ bJ & \bar{b}J \end{pmatrix} \neq 0, \quad \sigma(J) = \{\nu_1, \dots, \nu_m\}, \tag{3}$$

$$a_{11}b + (a_{12} + a_{21})bJ + a_{22}bJ^2 = 0. \tag{4}$$

The 1st order canonical system is closely related to the elliptic system (1)

$$\frac{\partial \phi}{\partial y} - J \frac{\partial \phi}{\partial x} = 0, \quad J = \text{diag} (J_1, \dots, J_m) \in \mathbb{C}^{l \times l}, \quad J_i \in \mathbb{C}^{l_i \times l_i}, \quad \sigma(J_i) = \{\nu_i\} \tag{5}$$

which, in particular, for  $J = i$  corresponds to the Cauchy-Riemann system.

For equation (1) in the half-plane  $D = \{y > 0\}$ , consider the boundary value problem

$$(pu_x + qu_y + p_0u)|_{\mathbb{R}} = g. \tag{6}$$

We search the solution  $u$  in the class  $C_{(\lambda)}^{1,\mu}(D; F)$ ,  $0 < \lambda < 1$ .

The right side  $g$  is selected from the class  $C_{\lambda-\sigma}^{\mu}(D; F)$ ,  $l \times l$ -the matrix coefficients  $p, q$  and  $p_0$  are assumed to be piecewise continuous on  $\mathbb{R}$  with possible discontinuities at points  $\tau \in F$ . More precisely, there exists a  $\varepsilon > 0$ , such that

$$p, q \in C_{(\varepsilon)}^{\mu+\varepsilon}(G; a, b), \quad p_0 \in C_{\varepsilon-\sigma}^{\mu+\varepsilon}(G; a, b) \tag{7}$$

for any interval  $G = (a, b)$ , into which the complement  $\mathbb{R} \setminus F_0$  is divided.

**Theorem 1.** *Problem (1), (6) in class  $\{u \in C_{(\lambda)}^{1,\mu}, u(\infty) = 0\}$ ,  $0 < \lambda < 1$ , is equivalent to finding analytic vector functions in this class  $\psi(z)$  by boundary condition*

$$\text{Re} (G\psi' + G_0\psi)|_{\mathbb{R}} = g \tag{8}$$

with matrix coefficients  $G = pb + qbJ$ ,  $G_0 = p_0b$ .

The connection between the solutions of these problems is carried out by the Bitsadze formula [1]

$$u = \text{Re } bE\psi, \tag{9}$$

where the operation  $E$  is given block by block by the equality

$$(E\psi)_i = E_i\psi_i = \sum_{k=0}^{l_i-1} \frac{y^k}{k!} (J_i - \nu_i)^k \psi_i^{(k)}(x + \nu_i y), \quad z = x + iy, \quad i = 1, \dots, m. \tag{10}$$



Let us consider the case when the coefficients  $p, q$  in (6) are constant, and  $p_0 = 0$ , i.e. the problem

$$(pu_x + qu_y)|_{\mathbb{R}} = g, \quad p, q \in \mathbb{R}^{l \times l}. \quad (11)$$

In this case, the matrix  $G = pb + qbJ \in \mathbb{C}^{l \times l}$  and the statements of Theorem 1 can be significantly supplemented.

**Theorem 2.** a) If  $\det G = 0$ , then homogeneous problem (1), (11) in the class  $C_{(\lambda)}^{1,\mu}$ ,  $0 < \lambda < 1$ , has an infinite number of linearly independent solutions. For example, if  $G\eta = 0$  for some nonzero  $\eta \in \mathbb{C}^l$ , then the solutions to the homogeneous problem are the functions  $u = \operatorname{Re} b(E\psi_0)\eta$  for any scalar analytic function  $\psi_0 \in C_{(\lambda)}^{1,\mu}$ .

b) If  $\det G \neq 0$ , then the condition

$$\int_{\mathbb{R}} g(t)dt = 0 \quad (12)$$

is necessary and sufficient for the solvability of an inhomogeneous problem in the class  $C_{(\lambda)}^{1,\mu}$ ,  $0 < \lambda < 1$ . Under the additional assumption  $u(\infty) = 0$  its solution is unique and is given by the formula

$$u(z) = \operatorname{Re} \frac{1}{\pi} \int_{\mathbb{R}} b[t - z]^{-1} G^{-1} f(t) dt, \quad f(x) = \int_{-\infty}^x g(t) dt. \quad (13)$$

We introduce the concept of the conjugate function  $v$  to the solution  $u$  of equation (1). It is determined by the line integral

$$v(z) = v(z_0) + \int_{z_0}^z (a_{21}u_x + a_{22}u_y)dx - (a_{11}u_x + a_{12}u_y)dy.$$

For  $v$  we get the representation

$$v = \operatorname{Re} c\phi, \quad c = a_{21}b + a_{22}bJ. \quad (14)$$

Together with  $u(z)$  this function also belongs to the class  $C_{(\lambda)}^{1,\mu}$ ,  $0 < \lambda < 1$ . If we put

$$\begin{pmatrix} b & \bar{b} \\ c & \bar{c} \end{pmatrix}^{-1} = \begin{pmatrix} b_2 & c_2 \\ \bar{b}_2 & \bar{c}_2 \end{pmatrix}$$

for its inverse matrix, we can write the formula

$$\phi = 2(b_2u + c_2v). \quad (15)$$

From (11) we can go to the “integrated” boundary value problem

$$(pu + qv)|_{\mathbb{R}} = f \quad (16)$$

in class  $C_{(\lambda)}^{1,\mu}$ ,  $0 < \lambda < 1$ , where  $p_1, q_1$  are again denoted by  $p, q$ .

As a simple example, consider for system (1) the boundary value problem

$$u(x, 0) + u(-x, 0) = f(x), \quad v(x, 0) + v(-x, 0) = g(x), \quad (17)$$

where  $f, g$  are given even  $l$ -vector functions.

**Theorem 3.** Problem (1), (17) in class  $u + iv \in C_{(\lambda)}^{\mu}(D, F)$ ,  $0 < \lambda < 1$ , is uniquely solvable and its solution is given by the formula

$$u(z) = \operatorname{Re} \frac{1}{\pi i} \int_{\mathbb{R}} b[t - z]^{-1} h(t) dt, \quad h = b_2f + c_2g, \quad (18)$$

where matrices  $b_2, c_2$  appear in (15).

In [2], the Fredholm solvability of boundary value problems for elliptic systems was studied. In [3], the Fredholm solvability of boundary value problems for high-order elliptic equations was studied and the index formula for the problem was calculated.

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**Keywords:** elliptic systems, boundary value problem, Fredholm solvability of the problem, problem index formula.

**2020 Mathematics Subject Classification:** 35J30, 35J40, 35J37

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## A periodic on part of variables solution of an infinite system in the broad sense

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The paper considers an infinite system of first order partial derivatives equations

$$Du = P(t, x, y)u + \mu f(t, x, y, u), \quad (1)$$

where  $D = \frac{\partial}{\partial t} + \langle a(t, x, y), \frac{\partial}{\partial x} \rangle + \langle b(t, x, y), \frac{\partial}{\partial y} \rangle$  -differentiation operator,

$\frac{\partial}{\partial x} = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right)$ ,  $\frac{\partial}{\partial y} = \left( \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k} \right)$  - differentiation vectors,  $x = (x_1, \dots, x_m)$  and  $a = (a_1, \dots, a_m)$ - $m$ -vectors,  $y = (y_1, \dots, y_k)$  and  $b = (b_1, \dots, b_k)$ - $k$ -vectors,  $\langle \cdot, \cdot \rangle$  - means the scalar product of this vectors,  $P = \{p_{ij}\}$ ,  $i = 1, 2, \dots, j = 1, 2, \dots$ - infinite matrix,  $u = (u_1, u_2, \dots)$ ,  $f = (f_1, f_2, \dots)$ - countable vectors,  $\mu$ -scalar parameter.

Known, that the classical solution  $u(t, x, y)$  of system (1) is continuous differentiable in all variables. If the solution  $u(t, x, y)$  has less smoothness, but in satisfies the system (1) in some sense, then it is called a generalized solution of the system (1).

In this paper, we construct a periodic in variables  $(t, x)$  solution of system (1) in the broad Friedrichs sense [1].

Note that the constructed solution in the broad sense of the system (1), requires only the continuity of the input data  $a$ ,  $b$ ,  $f$  and matrix  $P$ . If these input data have the desired smoothness, then the constructed solution in a broad sense turns into a classical solution of system (1).

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**Keywords:** first-order partial differential equations, periodic solutions, generalized solutions.

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## Weak Version of Symmetric Space

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Let  $E_i$  be a (quasi) symmetric Banach space on  $(0, 1)$ ,  $i = 1, 2$ . We define the pointwise product space  $E_1 \odot E_2$  as

$$(18) \quad E_1 \odot E_2 = \{f : f = f_1 f_2, f_i \in E_i, i = 1, 2\}$$

with a functional  $\|f\|_{E_1 \odot E_2}$  defined by

$$\|f\|_{E_1 \odot E_2} = \inf\{\|f_1\|_{E_1} \|f_2\|_{E_2} : f = f_1 f_2, f_i \in E_i, i = 1, 2\}.$$

**Definition 1.** Let  $E$  be a symmetric (quasi) Banach space on  $(0, 1)$ . The fundamental function  $\varphi_E$  is defined by  $\varphi_E(t) = \|\chi_A\|$ , where  $t \in [0, 1)$  and  $A$  is a measurable subset of  $(0, 1)$  with  $m(A) = t$ .

Let  $M_{\varphi_E}(0, 1)$  be the usual Marcinkiewicz spaces with norms defined by

$$M_{\varphi_E}(0, 1) = \{f \in L_0(0, 1) : \|f\|_{M_{\varphi_E}} = \sup_{t>0} \frac{\varphi_E(t)}{t} \int_0^t \mu_s(f) ds < \infty\}.$$

**Definition 2.** Let  $E$  be a symmetric (quasi) Banach space on  $(0, 1)$ . We call  $M_{\varphi_E}(0, 1)$  is weak version of  $E$  and denote it by  $E_\infty$ .

The classical weak  $L_p$ -space  $L_{p,inf ty}(0, 1)$  ( $1 \leq p < \infty$ ) is defined as the set of all measurable functions  $f$  on  $(0, 1)$  such that

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} \mu_t(f) < \infty.$$

For  $p > 1$ ,  $L_{p,\infty}(0, 1)$  can be renamed as a Banach space by

$$f \mapsto \sup_{t>0} t^{-1+\frac{1}{p}} \int_0^t \mu_s(f) ds.$$

If  $E = L_p(0, 1)$  ( $1 < p < \infty$ ), then  $E_\infty = L_{p,\infty}(0, 1)$ . If  $(L_p(0, 1))_\infty = L_p(0, 1)$ . But for  $0 < p \leq 1$ , if  $f \in (L_p(0, 1))_\infty$ , then

$$\|f\|_{(L_p(0,1))_\infty} = \sup_{t>0} t^{\frac{1}{p}-1} \int_0^t \mu_s(f) ds = \int_0^1 \mu_s(f) ds = \|f\|_1.$$

Hence,  $(L_p(0, 1))_\infty = L_1(0, 1)$  and it is different from the classical weak  $L_p$ -space.

Let  $\Phi$  be an N-function, we define

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

If  $b_\Phi < \infty$ , then the fundamental function of Orlicz space  $L_\Phi(0, 1)$  on  $(0, 1)$  equipped with the Luxemburg norm, is the following

$$\varphi_{L_\Phi(\Omega)}(t) = 1/\Phi^{-1}(\frac{1}{t}), \quad t > 0.$$

Hence, if  $E = L_\Phi(0, 1)$  and  $1 < a_\Phi \leq b_\Phi < \infty$ , then  $E_\infty = L_{\Phi,\infty}(0, 1)$ .

**Proposition 1.** Let  $E_i$  be symmetric (quasi) Banach space on  $(0, 1)$  which is  $\alpha_i$ -convex for some  $0 < \alpha_i < \infty$  ( $i = 1, 2$ ). Then  $E_1$  and  $E_2$  can be equipped with equivalent quasi norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ , respectively, so that  $\varphi_{E_1 \odot E_2}(t) = \varphi_{E_1}(t)\varphi_{E_2}(t)$ , for any  $t \geq 0$ .

**Theorem 1.** Let  $E_i$  be symmetric (quasi) Banach space on  $(0, 1)$  which is  $\alpha_i$ -convex for some  $0 < \alpha_i < \infty$  ( $i = 1, 2$ ) and  $0 < a < 1$ . If  $x \in ((E_1^{(a)})_\infty)^{(\frac{1}{a})}(\mathcal{M})$  and  $y \in ((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}(\mathcal{M})$ , then  $xy \in (E_1 \odot E_2)_\infty(\mathcal{M})$  and the following Holder type inequality holds

$$\|xy\|_{(E_1 \odot E_2)_\infty} \leq \|x\|_{((E_1^{(a)})_\infty)^{(\frac{1}{a})}} \|y\|_{((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}}.$$

**Theorem 2.** *Let  $E$  be symmetric (quasi) Banach space on  $(0, 1)$ . Then we have the following Chebyshev type inequality*

$${}^t\varphi_E(\tau(e_{(t,\infty)}(|x|))) \leq \|x\|_{E_\infty}, \quad \forall x \in E_\infty(\mathcal{M}).$$

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# Initial-Boundary Value Problems for a Two-Dimensional Wave Equation with Nonlocal Conditions which are Multidimensional Generalizations of the Samarskii-Ionkin Problem

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It is well known that initial boundary value problems for the wave equation often arise in the context of physical systems that describe the propagation of waves, such as sound waves, water waves, electromagnetic waves, and others. These problems typically involve a combination of initial conditions and boundary conditions for a differential equation.

Problems with nonlocal boundary conditions represent a special class of initial boundary value problems in which the boundary conditions simultaneously connect the values of the desired solution (and its derivatives) at different points of the boundary. Such conditions can occur in a variety of physical and mathematical contexts and can describe more complex physical phenomena and boundary conditions.

The report discusses the formulation of new initial-boundary value problems for a two-dimensional wave equation with nonlocal conditions respect to spatial variables, which are multidimensional generalizations of the Samarsky-Ionkin problem. The domain of consideration of the problem is a circular cylinder  $Q$  with its axis along the  $t$  axis. Classical initial conditions are set at the base of the cylinder and new nonlocal boundary conditions on the spatial (lateral) boundaries of the cylinder.

Let  $\Omega = \{(r, \varphi) : 0 \leq r < 1, 0 \leq \varphi \leq 2\pi\}$  be the unit circle,  $Q = \{(r, \varphi, t) : 0 \leq r < 1, 0 \leq \varphi \leq 2\pi, 0 < t < T\}$  is a right circular cylinder.

We will consider a new nonlocal boundary value problem for the two-dimensional wave equation:

$$u_{tt}(r, \varphi, t) - \Delta u(r, \varphi, t) = f(r, \varphi, t), \quad (r, \varphi, t) \in Q, \tag{1}$$

where  $\Delta$  is the Laplace operator in polar coordinates  $(r, \varphi)$ .

We will use classical initial conditions

$$u|_{t=0} = \tau(r, \varphi), \quad u_t|_{t=0} = \nu(r, \varphi), \quad (r, \varphi) \in \Omega, \tag{2}$$

and nonlocal boundary conditions on the lateral boundary of a circular cylinder

$$u(1, \varphi, t) - \alpha u(1, 2\pi - \varphi, t) = 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq t \leq T, \tag{3}$$

$$\frac{\partial u}{\partial r}(1, \varphi, t) - \frac{\partial u}{\partial r}(1, 2\pi - \varphi, t) = 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq t \leq T. \tag{4}$$

Here  $\alpha \neq 1$  is a fixed real number.

We take the right side of the equation and the initial conditions from the following “standard” smoothness class for hyperbolic problems:  $f(r, \varphi, t) \in C^{1+\epsilon}(\bar{Q})$ ;  $\tau(r, \varphi) \in C^{2+\epsilon}(\bar{\Omega})$ ;  $\nu(r, \varphi) \in C^{1+\epsilon}(\bar{\Omega})$ . Additionally, we require that  $\tau(r, \varphi)$  and  $\nu(r, \varphi)$  satisfy boundary conditions (3), (4).

To solve the initial-boundary-value problem (1)-(4), we apply the method of reducing to a sequential solution of two initial-boundary value problems with self-adjoint boundary conditions in the spatial variable, proposed in [1] for the case of one-dimensional parabolic initial-boundary value problems with non-reinforced regular boundary value problems conditions.

The main result of the work is the proof of the correctness of the formulated problem in the classical sense.

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**Keywords:** multidimensional wave equation, nonlocal boundary condition, Samarskii-Ionkin condition, classical solution.

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## On the Compactness of the Commutator of the Riesz Potential in Global Morrey Type Spaces

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In this work a sufficient conditions for the boundedness and compactness of the commutator of the Riesz Potential in global Morrey-type spaces are obtained.

DEFINITION. Let  $0 < p, \theta \leq \infty$  and let  $w$  be a nonnegative measurable function on  $(0, \infty)$ . The general global Morrey-type space  $GM_{p\theta, w(\cdot)} \equiv GM_{p\theta, w(\cdot)}(\mathbb{R}^n)$  is defined as the set of all functions  $f \in L^1_{loc}(\mathbb{R}^n)$  with finite quasi-norm

$$\|f\|_{GM_{p\theta, w(\cdot)}} \equiv \sup_{x \in \mathbb{R}^n} \left\| w(r) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0, \infty)},$$

where  $B(t, r)$  the ball with center at the point  $t$  and of radius  $r$ .

We denote by  $\Omega_{p\theta}$  the set of all functions that are nonnegative, measurable on  $(0, \infty)$ , not equivalent to 0 and such, that for some  $t > 0$  (and therefore for all  $t > 0$ )

$$\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} < \infty, \quad \|w(r)\|_{L_\theta(t, \infty)} < \infty.$$

The space  $GM_{p\theta, w(\cdot)}$  is non-trivial, that is, it consists not only of functions equivalent to 0 in  $\mathbb{R}^n$  if and only if  $w \in \Omega_{p\theta}$ .

We consider the Riesz Potential  $I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n$ .

For a function  $b \in L^1_{loc}(\mathbb{R}^n)$  by  $M_b$  denote multiplier operator  $M_b f = bf$ , where  $f$ -measurable function. Then the commutator between  $I_\alpha$  and  $M_b$  is defined by

$$[b, I_\alpha] = M_b I_\alpha - I_\alpha M_b = \int_{\mathbb{R}^n} \frac{(b(x) - b(y)) f(y)}{|x-y|^{n-\alpha}} dy.$$

Let  $u, v$  be weight functions. Denote by

$$H^* g(t) := \int_t^\infty g(s) ds, \quad g \in \mathfrak{M}^+,$$

the Hardy operator,

$$W(t) := \int_0^t w(t) dw,$$

$$U_*(t) := \int_t^\infty u(t) du,$$

$$V_*(t) := \int_t^\infty v(t) dv.$$

**Theorem 1.** Let  $1 < p \leq q < \infty, 0 < \alpha < n$  and  $b \in BMO(\mathbb{R}^n), 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, w_1, w_2 \in \Omega_\theta$  and let the functions  $w_1, w_2$  satisfy the conditions

$$A_0^* := \sup_{t>0} \left( \int_t^\infty \int_\tau^\infty (1 + \ln \frac{\tau}{r}) dr w_1(\tau) d\tau \right)^{\frac{1}{q}} \left( \int_t^\infty v(t) dv \right)^{-\frac{1}{p}} < \infty,$$

$$A_1^* := \sup_{t>0} W_2^{\frac{1}{q}}(t) \left( \int_t^\infty \left( \frac{U_*(\tau)}{V_*(\tau)} \right)^{p'} v(\tau) d\tau \right)^{\frac{1}{p'}} < \infty.$$

Then the commutator  $[b, I_\alpha]$  is the boundedness operator from  $GM_{p\theta}^{w_1}$  to  $GM_{q\theta}^{w_2}$ .



**Theorem 2.** Let  $1 < p \leq q < \infty$ ,  $0 < \alpha < n$  and  $b \in VMO(\mathbb{R}^n)$ ,  $1 < p < \frac{n}{\alpha}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ ,  $w_1, w_2 \in \Omega_\theta$  satisfy the conditions of Theorem 1.

Then the commutator  $[b, I_\alpha]$  is a compact operator from  $GM_{p\theta}^{w_1}$  to  $GM_{q\theta}^{w_2}$ .

REMARK. The compactness of the commutator of the Riesz potential in Morrey space was considered in [1]. The compactness of sets in global Morrey-type spaces was studied in [2], [3].

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# Inverse Source Problem for Time-Fractional Diffusion Equation with Nonlocal Boundary Condition

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In this paper, we consider the inverse problem of finding a pair of functions  $\{p(t), u(x, t)\}$  such that it satisfies the equation

$${}_0\partial_t^\alpha u = u_{xx} + p(t)f(x, t), \quad (x, t) \in D_T, \tag{1}$$

the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \tag{2}$$

the boundary conditions

$$u(0, t) = 0, \quad u_x(0, t) + du_{xx}(1, t) = 0, \quad 0 \leq t \leq T, \tag{3}$$

and the overdetermination condition

$$\int_0^1 u(x, t) dx = E(t), \quad t \in [0, T], \tag{4}$$

where  $T > 0$ ,  $D_T = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$ ,  $f, \varphi$  are given real-valued functions,  $d > 0$  is a real constant,  ${}_0\partial_t^\alpha u$  is the Caputo time-fractional derivative of order  $\alpha$  defined in [1] by

$${}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1,$$

and  $\Gamma(\cdot)$  is the Gamma function.

In the case when  $\alpha = 1$ , the inverse problem of finding pairs  $\{p(t), u(x, t)\}$  with the conditions (2)-(4) for the equation

$$u_t = u_{xx} + p(t)u + f(x, t), \quad (x, t) \in D_T$$

is considered in [2].

The well-posedness of the inverse problem (1)-(4) is shown by Fourier expansion in terms of eigenfunctions of the spectral problem, which has a spectral parameter in the boundary condition. Also, for the well-posedness of the problem (1)-(4), the properties of the Volterra integral equation of the second kind have been used. One of the main results of this work is that the existence and uniqueness of the solution of the problem (1)-(4) has been proved without using the orthogonality condition on the input data. The continuous dependence of the solution of the problem (1)-(4) on the data has been proved.

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**Keywords:** inverse source problem, fractional diffusion equation, nonlocal boundary condition, problem with spectral parameter in boundary condition, Mittag-Leffler type function.

**2020 Mathematics Subject Classification:** 35K20, 35R30, 35A09

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## On a Nonlocal Frankl-Type Boundary Value Problem for an Equation of Mixed Parabolic-Hyperbolic Type

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In this talk a new nonlocal boundary value problem for an equation of the mixed type is formulated. This equation is parabolic-hyperbolic and belongs to the first kind because the line of type change is not a characteristic of the equation. Nonlocal condition binds points on boundaries of the parabolic and hyperbolic parts of the domain with each other. This problem is generalization of the well-known problems of Frankl type. A boundary value problem for the heat equation with conditions of the Samarskii-Ionlin type arises in solving this problem. Unlike the existing publications of the other authors related to the theme it is necessary to note that in this papers the nonlocal problems were considered in rectangular domains. But in our formulation of the problem the hyperbolic part of the domain coincides with a characteristic triangle. Unique strong solvability of the formulated problem is proved.

Let  $\Omega \subset R^2$  be a finite domain bounded for  $y > 0$  by the segments  $AA_0$ ,  $A_0B_0$ ,  $B_0B$ ,  $A = (0, 1)$ ,  $B_0 = (1, 1)$ ,  $B = (1, 0)$  and for  $y < 0$  by the characteristics  $AC : x + y = 0$  and  $BC : x - y = 1$  of an equation of the mixed parabolic-hyperbolic type

$$Lu = \begin{cases} u_x - u_{yy}, y > 0 \\ u_{xx} - u_{yy}, y < 0 \end{cases} = f(x, y). \quad (1)$$

This is the equation of the mixed type. The equation refers to the first kind because the line of change of type  $y = 0$  is not a characteristic of the equation.

By  $W_2^1(\Omega) = H^1(\Omega)$  we denote the space of S.L. Sobolev with the scalar product  $(\cdot, \cdot)_l$  and the norm  $\|\cdot\|_l$ ,  $W_2^0(\Omega) = L_2(\Omega)$ ;  $\Omega_1 = \Omega \cap \{y > 0\}$ ,  $\Omega_2 = \Omega \cap \{y < 0\}$ .

In  $\Omega$  consider the following nonlocal boundary value problem being the generalization of an analogue of the Frankl problem for the parabolic-hyperbolic equation (1).

**Problem F.** *Find a solution to Eq. (1) satisfying classical boundary conditions*

$$u|_{AA_0} = 0, \quad u_y|_{A_0B_0} = 0 \quad (2)$$

*and a nonlocal boundary condition*

$$u(\theta(t)) = au(\theta_0(t)) + bu(\theta_1(t)), \quad 0 \leq t \leq 1, \quad (3)$$

*where  $\theta(t) = (t, 1)$ ,  $\theta_0(t) = (\frac{t}{2}, -\frac{t}{2})$ ,  $\theta_1(t) = (\frac{t+1}{2}, \frac{t-1}{2})$ ;  $a, b$  are arbitrary numbers.*

It is easy to see that  $\theta(t) \in A_0B_0$ ,  $\theta_0(t) \in AC$ ,  $\theta_1(t) \in BC$ . Therefore the new nonlocal boundary condition (3) binds with each other values of the sought-for solution on the parabolic part of the boundary  $A_0B_0$  and on the hyperbolic parts of the boundary of the domain (at the characteristics  $AC$  and  $BC$ ).

**Theorem.** *Let  $a + b \neq 0$ . Then for any function  $f \in L_2(\Omega)$  there exists a unique strong solution  $u(x, y)$  to the problem F. This strong solution belongs to the class  $H^1(\Omega) \cap H_{x,y}^{1,2}(\Omega_1) \cap C(\bar{\Omega})$ , and satisfies the inequality*

$$\|u\|_1 \leq C \|f\|_0.$$

Note that in the special case when  $a = 1 + \alpha$  and  $b = 1 - \alpha$  ( $\alpha$  is a real number), the problem F was considered in our paper [1].

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**Keywords:** equation of the mixed type, nonlocal boundary condition, Frankl problem, strong solution, conditions of the Samarskii-Ionlin type.

**2020 Mathematics Subject Classification:** 35M12

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## Solvable Problems for the Laplace-Beltrami Operator on a Two-Dimensional Sphere with a Cut

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It is known [1] that the two-dimensional sphere  $S$  in the three-dimensional space  $\mathbb{R}^3$  represents a Riemannian manifold. The Laplace-Beltrami operator  $\Delta_S$  on the indicated sphere in a standard way is introduced. Since the sphere  $S$  is a smooth manifold without boundary, the problem is well defined.

$$u(x) - \Delta_S u(x) = f(x), \quad x \in S.$$

The sphere  $S$  of the closed curve  $C$  is split into two non-intersecting parts  $S_2$  and  $S_1$  [2],[3].

The following Dirichlet problem studied in the work [2]

$$-\Delta_S u(x) = 0, \quad x \in S_2, \quad u(x) = g(x), \quad x \in C.$$

Using the potentials of a simple and double layer, the existence of a solution to the indicated Dirichlet problem is proved.

Another problem is studied in the work [3]

$$\begin{aligned} -\Delta_S u(x) &= \omega(x), \quad x \in S_2, & (1) \\ -\frac{1}{2}u(x) + \int_C u(y) \underline{curl}_S \varepsilon(x, y) \cdot \vec{i}(y) ds_y - \\ - \int_C \varepsilon(x, y) \underline{curl}_S u(y) \cdot \vec{i}(y) ds_y &= 0, \quad x \in C. & (2) \end{aligned}$$

Here  $\varepsilon(x, y) = -\frac{1}{4\pi} \ln |1 - \langle x, y \rangle|$  represents the fundamental solution of the Laplace-Beltrami operator. It is shown in the work [3], that the solution to problem (1),(2) is written as

$$u(x) = \int_{S_2} \varepsilon(x, y) \omega(y) d\sigma_y, \quad x \in S_2.$$

We also note the paper [4], where a boundary value problem for the Laplace-Beltrami operator on a punctured sphere was studied. A punctured sphere is a sphere from which one point has been removed. In this case, the problem arises: What additional conditions must the solution at the remote point satisfy in order to guarantee the uniqueness of the solution?

In the present paper, a fixed arc from the sphere is removed and the same question is studied: what additional conditions must the solution on the removed arc satisfy in order to guarantee the uniqueness of such a solution?

To solve this problem we introduce a class of the function  $W_{2,loc}^2(S_{20}) = \bigcup_{\delta > 0} W_2^2(S_{2\delta})$ .

As well as necessary to introduce a class of the function

$$\begin{aligned} W_{2L}^2(S_{20}) &= \{h \in W_{2,loc}^2(S_{20}) : \lim_{\delta \rightarrow 0} \int_{S_{20}} h(y) d\sigma_y = 0, \\ (L_d h)(x) - (L_s h)(x) &\in H(S_{20}), \Delta_S h \in L_2(S)\}. \end{aligned}$$

Here  $L_s h$  and  $L_d h$  are single and double layer potentials respectively,  $h(x)$  is an arbitrary function from the class  $W_{2L}^2(S_{20})$ . Since  $\Delta_S h \in L_2(S)$ , that  $T(x) \in W_2^2(S)$

and  $\Delta_S T = \Delta_S h$ . Thus,  $T(x)$  represents a regularization of the function  $h(x)$ . In fact, the function  $h(x)$  could have singularities on the curve  $C$ . At the same time, its regularization  $T(x)$  no longer has singularities on the full sphere  $S$ , that is, it belongs to the space  $W_2^2(S)$ .

**Lemma 1.** *An arbitrary element  $h$  of the class  $W_{2L}^2(S_{20})$  for  $x \in S_{20}$  can be represented as  $h(x) = T(x) + (L_s h)(x) - (L_d h)(x)$ , where  $T(x)$  is a function from  $W_2^2(S)$ .*

Now we can formulate the statement.

**Theorem 1.** *For any function  $f \in L_2(S)$  and any function  $g$  from  $H(S_{20})$ , the problem*

$$\begin{aligned} -\Delta_S u(x) &= f(x), & x \in S_{20}, \\ (L_d u)(x) - (L_s u)(x) &= g(x), & x \in S_{20} \end{aligned}$$

*has a unique solution in the class  $W_{2L}^2(S_{20})$ .*

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**Keywords:** Laplace-Beltrami operator, single and double layer potentials, two-dimensional sphere.

**2020 Mathematics Subject Classification:** 31A15, 47B40, 31C12

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# Global Existence and Blow-up of Solutions to Porous Medium Equation and Pseudo-Parabolic Equation Related to Baouendi-Grushin Operator

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In this talk, we discuss a global existence and blow-up of the positive solutions to the initial-boundary value problem of the nonlinear porous medium equation and the nonlinear pseudo-parabolic equation related to the Baouendi-Grushin operator. Our approach is based on the Poincaré inequality from [1] for the Baouendi-Grushin vector fields and the concavity argument.

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**Keywords:** Blow-up, porous medium equation, global solution, pseudo-parabolic equation, Baouendi-Grushin operator.

**2010 Mathematics Subject Classification:** 35K91, 35B44, 35A01

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## On Pantograph Delay Differential Equations

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In this talk, we prove by means of some fixed point theorems, the existence and uniqueness of solutions for a class of fractional pantograph delay differential equations with initial conditions. Then, we investigate the stability and Ulam-Hyers stability for this type of delay differential equation. Furthermore, we apply the Adomian decomposition method to analyze numerically some examples.

**Keywords:** Fixed point Theorem, Fractional derivative, initial condition, Pantograph equation, Adomian decomposition method.

**2020 Mathematics Subject Classification:** 26A33, 34A08, 34B15.

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## Construction of an Unconditional Basis from Elements of the System of Root Functions of one Spectral Problem with a Linear Occurrence of the Spectral Parameter in the Boundary Value Condition

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It is well known that system of eigenfunctions of an operator defined formally by a self-adjoint differential expression with arbitrary self-adjoint boundary value conditions, providing a discrete spectrum, forms an orthogonal basis of the space  $L_2$ . In the case when the spectral parameter is also present in the boundary value condition, consideration of the problem in the form of an operator in  $L_2$  becomes impossible, since the spectral parameter should not change the domain of the operator. Problems of this type naturally arise when we solve (by the method of separation of variables) initial-boundary value problems for evolutionary equations, when the boundary value condition includes derivatives of the solution with respect to the time variable. For such problems, based on the theory developed by A.A. Shkalikov [1], it is known that when we remove (some finite number of) elements, the system of root vectors of such a problem can already form a basis with brackets. However, the question of whether the system forms an unconditional basis must be investigated for each problem separately. We consider a spectral problem with a linear occurrence of the spectral parameter in one of the boundary conditions

$$l(u) \equiv -u''(x) = \lambda u(x), \quad 0 < x < 1, \quad (1)$$

$$u'(0) = 0, \quad u'(1) = \lambda(u(0) + u(1)). \quad (2)$$

Representing the general solution of the equation (1) by the formula:

$$u(x, \lambda) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x$$

and satisfying it by the boundary value conditions (2), we obtain a linear system with respect to the coefficients  $C_1$  and  $C_2$ :

$$\begin{cases} C_2 = 0, \\ C_1 \left( \sqrt{\lambda} \left[ \sqrt{\lambda}(1 + \cos \sqrt{\lambda}) + \sin \sqrt{\lambda} \right] \right) = 0. \end{cases} \quad (3)$$

Therefore, the characteristic determinant of the problem (1), (2) has the form

$$\Delta(\lambda) = \sqrt{\lambda}(1 + \cos \sqrt{\lambda}) + \sin \sqrt{\lambda} = \cos \frac{\sqrt{\lambda}}{2} \left( \sqrt{\lambda} \cos \frac{\sqrt{\lambda}}{2} + \sin \frac{\sqrt{\lambda}}{2} \right).$$

Solving the equation  $\Delta(\lambda) = 0$ , we have two series of eigenvalues

$$\cos \frac{\sqrt{\lambda}}{2} = 0 \Rightarrow \sqrt{\lambda_k} = (2k+1)\pi \Rightarrow \lambda_k^{(1)} = ((2k+1)\pi)^2;$$

$$\sqrt{\lambda} \cos \frac{\sqrt{\lambda}}{2} + \sin \frac{\sqrt{\lambda}}{2} = 0 \Leftrightarrow \cot \frac{\sqrt{\lambda}}{2} = -\frac{1}{\sqrt{\lambda}}.$$

Due to the Rouché theorem, we get

$$\frac{\sqrt{\lambda}}{2} = \left( \frac{\pi}{2} + k\pi \right) + \frac{\delta_k}{2} \Rightarrow \sqrt{\lambda_k} = (2k+1)\pi + \delta_k,$$

$\delta_k$  are bounded. Let us prove that  $\delta_k \rightarrow 0$ .

$$\cot \left( \frac{\pi}{2} + k\pi + \frac{\delta_k}{2} \right) = -\frac{1}{(2k+1)\pi + \delta_k},$$

$$\begin{aligned} \cot \left[ \left( \frac{\pi}{2} + k\pi \right) + \frac{\delta_k}{2} \right] &= \frac{\cot \left( \frac{\pi}{2} + k\pi \right) \cdot \cot \frac{\delta_k}{2} - 1}{\cot \left( \frac{\pi}{2} + k\pi \right) + \cot \frac{\delta_k}{2}} = \frac{-1}{\cot \frac{\delta_k}{2}} = \frac{1}{(2k+1)\pi + \delta_k} \Leftrightarrow \\ &\Leftrightarrow \tan \frac{\delta_k}{2} = \frac{1}{(2k+1)\pi + \delta_k} = O \left( \frac{1}{k} \right) \Rightarrow \\ \Rightarrow \delta_k &= O \left( \frac{1}{k} \right). \text{ Thus, } \sqrt{\lambda_k} = (2k+1)\pi + O \left( \frac{1}{k} \right), \text{ that is } \lambda_k^{(1)} = ((2k+1)\pi + \delta_k)^2. \end{aligned}$$

$\lambda_k^{(1)}, \lambda_k^{(2)}$  are simple (single) eigenvalues of the problem (1)-(2). From the system of equations (3):  $C_2 = 0$ , therefore, from the general solution of the equation (1), the system of eigenfunctions of the problem (1)-(2) is explicitly calculated and has the form:

$$\begin{aligned} u_k^{(1)}(x) &= \cos((2k+1)\pi)(x), \\ u_k^{(2)}(x) &= \cos((2k+1)\pi + \delta_k)(x), \quad k = 0, 1, 2, 3, \dots, \text{ where, } \delta_k = O \left( \frac{1}{k} \right). \end{aligned} \quad (4)$$

Due to the asymptotics of  $\delta_k$ , by removing one element from this system, we obtain the complete and minimal system in  $L_2(0, 1)$ . For large enough  $k \gg 1$ , this system does not form an unconditional basis, since two series of eigenfunctions  $u_k^{(1)}(x)$  and  $u_k^{(2)}(x)$  "stick together", although according to the results of A.A. Shkalikov [1], this system forms a Riesz basis with brackets in  $L_2(0, 1)$ .

We introduce the following auxiliary system of functions:

$$\begin{aligned} y_k^{(1)}(x) &= u_k^{(1)}(x), \\ y_k^{(2)}(x) &= \frac{1}{\delta_k} (u_k^{(2)}(x) - u_k^{(1)}(x)), \quad k = 0, 1, 2, 3, \dots, \end{aligned} \quad (5)$$

that is, the auxiliary system of functions has the form

$$\begin{aligned} y_k^{(1)}(x) &= \cos(2k+1)\pi x, \\ y_k^{(2)}(x) &= -\frac{\sin(\delta_k x)}{\delta_k} \sin(2k+1)\pi x + \frac{\cos(\delta_k x) - 1}{\delta_k} \cos(2k+1)\pi x, \quad k = 0, 1, 2, 3, \dots \end{aligned} \quad (6)$$

We consider the system of functions

$$\{\varphi_k^{(1)}(x) = \cos(2k+1)\pi x, \varphi_k^{(2)}(x) = -x \sin(2k+1)\pi x\}_{k=0}^{\infty}. \quad (7)$$

These functions are eigenfunctions ( $\varphi_k^{(1)}(x)$ ) and adjoint functions ( $\varphi_k^{(2)}(x)$ ) of the Samarsky-Ionkin type problem:

$$\begin{aligned} -\varphi''(x) &= \lambda\varphi(x), \quad 0 < x < 1, \\ \varphi(0) + \varphi(1) &= 0, \quad \varphi'(1) = 0. \end{aligned}$$

As shown in [2], the system of eigenfunctions and associated functions of this problem forms the Riesz basis in  $L_2(0, 1)$ .

Let us show that the system of functions  $\{y_k^{(1)}(x), y_k^{(2)}(x)\}$  and  $\{\varphi_k^{(1)}(x), \varphi_k^{(2)}(x)\}$  are quadratically close.

Indeed, we get:

$$\sum_{k=1}^{\infty} \|\varphi_k - y_k\|^2 = \sum_{k=1}^{\infty} \|\varphi_k^{(2)} - y_k^{(2)}\|^2.$$

By direct calculation, we obtain the estimation:

$$\begin{aligned} \|\varphi_k^{(2)} - y_k^{(2)}\| &= \\ &= \left\| -x \sin(2k+1)\pi x + \frac{\sin(\delta_k x)}{\delta_k} \sin(2k+1)\pi x - \frac{\cos(\delta_k x) - 1}{\delta_k} \cos(2k+1)\pi x \right\| = \\ &= \left\| \sin(2k+1)\pi x \left( \frac{\sin(\delta_k x)}{\delta_k} - x \right) - \frac{\cos(\delta_k x) - 1}{\delta_k} \cos(2k+1)\pi x \right\| \leq \end{aligned}$$

$$\begin{aligned} &\leq \left\| \frac{\sin(\delta_k x)}{\delta_k} - x \right\| + \left\| \frac{1 - \cos(\delta_k x)}{\delta_k} \right\| = \\ &\left\| \frac{1}{\delta_k} \left( \delta_k x - \frac{\delta_k^3 x^3}{3!} + O(\delta_k^5 x^5) \right) - x \right\| + \left\| \frac{1}{\delta_k} \left( 1 - 1 + \frac{\delta_k^2 x^2}{2!} - O(\delta_k^4 x^4) \right) \right\| = \\ &= \left\| x - x - \frac{\delta_k^2 x^3}{3!} + O(\delta_k^4 x^5) \right\| + \left\| \delta_k \frac{x^2}{2!} - O(\delta_k^3 x^4) \right\| \leq O(\delta_k). \end{aligned}$$

Taking into account the asymptotics of  $\delta_k = O\left(\frac{1}{k}\right)$ , we obtain

$$\sum_{k=1}^{\infty} \|\varphi_k - y_k\|^2 = \sum_{k=1}^{\infty} \|\varphi_k^{(2)} - y_k^{(2)}\|^2 \leq C \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

that is the system of functions  $\{y_k^{(1)}(x), y_k^{(2)}(x)\}$  and  $\{\varphi_k^{(1)}(x), \varphi_k^{(2)}(x)\}$  are quadratically close.

Thus, we have proved

**Theorem.** *The system of eigenfunctions (4) of the spectral problem (1)-(2) does not form an unconditional basis in  $L_2(0, 1)$ . The auxiliary system of functions (5) forms an unconditional basis in  $L_2(0, 1)$ .*

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**Keywords:** Boundary value conditions, second order differential operator, Riesz basis, eigenfunctions, spectral parameter, characteristic determinant.

**2020 Mathematics Subject Classification:** 35M10, 35M20

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# Frankl-Type Nonlocal Boundary Value Problem for one Parabolic-Hyperbolic Equation

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The talk discusses a new type of boundary value problem involving an equation of mixed type, specifically one that is parabolic-hyperbolic and belongs to the first kind due to the absence of characteristic lines. The equation under consideration is of mixed type, meaning it combines characteristics of both parabolic and hyperbolic equations. Such equations are characterized by different types of behavior in different regions of the domain. This problem involves nonlocal boundary conditions. Nonlocal conditions are those that involve relationships between points on the boundaries of different parts of the domain. In this case, it appears to relate points on the boundaries of the parabolic and hyperbolic parts of the domain. The problem discussed in the talk is a generalization of problems of Frankl type. Frankl type problems are a specific class of mathematical problems that typically involve boundary value problems for partial differential equations. Unlike previous publications, which often considered nonlocal problems in rectangular domains, this talk's problem formulation involves a domain where the hyperbolic part coincides with a characteristic triangle. The talk concludes by claiming the unique strong solvability of the formulated problem. This means that there exists a unique solution that is well-behaved and satisfies the given conditions. In summary, the talk introduces and discusses a novel boundary value problem involving a mixed-type equation with nonlocal boundary conditions.

Let  $\Omega \subset R^2$  be a finite domain bounded for  $y > 0$  by the segments  $AA_0, A_0B_0, B_0B$ ,  $A = (0, 1), B_0 = (1, 1), B = (1, 0)$  and for  $y < 0$  by the characteristics  $AC : x + y = 0$  and  $BC : x - y = 1$  of an equation of the mixed parabolic-hyperbolic type

$$Lu = \begin{cases} u_x - u_{yy}, y > 0 \\ u_{xx} - u_{yy}, y < 0 \end{cases} = f(x, y). \tag{1}$$

This is the equation of the mixed type. The equation refers to the first kind because the line of change of type  $y = 0$  is not a characteristic of the equation.

By  $W_2^l(\Omega) = H^l(\Omega)$  we denote the space of S.L. Sobolev with the scalar product  $(\cdot, \cdot)_l$  and the norm  $\|\cdot\|_l, W_2^0(\Omega) = L_2(\Omega); \Omega_1 = \Omega \cap \{y > 0\}, \Omega_2 = \Omega \cap \{y < 0\}$ .

In  $\Omega$  consider the following nonlocal boundary value problem being the generalization of an analogue of the Frankl problem for the parabolic-hyperbolic equation (1).

**Problem F.** Find a solution to Eq. (1) satisfying classical boundary conditions

$$u|_{AA_0} = 0, \quad u_y|_{A_0B_0} = 0 \tag{2}$$

and a nonlocal boundary condition

$$\beta u(\theta(t)) = (1 + \alpha)u(\theta_0(t)) + (1 - \alpha)u(\theta_1(t)), \quad 0 \leq t \leq 1, \tag{3}$$

where  $\theta(t) = (t, 1), \theta_0(t) = (\frac{t}{2}, -\frac{t}{2}), \theta_1(t) = (\frac{t+1}{2}, \frac{t-1}{2}); \alpha, \beta$  are arbitrary numbers.

It is easy to see that  $\theta(t) \in A_0B_0, \theta_0(t) \in AC, \theta_1(t) \in BC$ . Therefore the new nonlocal boundary condition (3) binds with each other values of the sought-for solution on the parabolic part of the boundary  $A_0B_0$  and on the hyperbolic parts of the boundary of the domain (at the characteristics  $AC$  and  $BC$ ).

**Theorem.** Let  $\beta \neq 0$ . Then for any function  $f \in L_2(\Omega)$  there exists a unique strong solution  $u(x, y)$  to the problem  $F$ . This strong solution belongs to the class  $H^1(\Omega) \cap H_{x,y}^{1,2}(\Omega_1) \cap C(\bar{\Omega})$ , and satisfies the inequality

$$\|u\|_1 \leq C \|f\|_0.$$

Note that in the special case when  $\beta = 1$ , the problem  $F$  was considered in paper of G.Dildabek [1].

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**Keywords:** equation of the mixed type, nonlocal boundary condition, Frankl problem, strong solution, conditions of the Samarskii-Ionlin type.

**2020 Mathematics Subject Classification:** 35M12

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# On the Regularity of the Solution of a Hyperbolic Equation

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Let  $\Omega$  be a bounded domain in  $R^n$  with a sufficiently smooth boundary  $\partial\Omega$ . We discuss questions about the regularity of the solution the initial-boundary problem

$$\partial_t(t^\beta \partial_t u) - \Delta u = f \text{ in } Q = \Omega \times (0, T), \tag{1}$$

$$u = 0 \text{ on } \Sigma = \partial\Omega \times (0, T), \tag{2}$$

$$u(x, 0) = 0, \quad \lim_{t \rightarrow +0} t^\beta \partial_t u(x, t) = 0 \text{ in } \Omega, \tag{3}$$

which was studied in the dissertation of N. Kaharman [1]. In particular, they established the following result:

**Theorem 1.** *Let  $\beta \in [0, 1)$ ,  $f \in L^2(Q)$ ,  $(-\Delta)^{1-\nu} f \in L^2(Q)$ . Then problem (1)–(3) is uniquely solvable, and there is an a priori estimate*

$$\begin{aligned} \|u\|_{W_{2,t^\beta}^{2,2}(Q)}^2 &\equiv \|t^\beta \partial_t u\|_{W_2^1(0,T;L^2(\Omega))}^2 + \|u\|_{L^2(0,T;W_2^2(\Omega) \cap \overset{\circ}{W}_2^{\frac{1}{2}}(\Omega))}^2 \\ &\leq C \left[ \|f\|_{L^2(Q)}^2 + \|(-\Delta)^{1-\nu} f\|_{L^2(Q)}^2 \right], \text{ where } \nu = \frac{1-\beta}{2-\beta}, \end{aligned} \tag{4}$$

*i.e. the parameter  $\nu$  changes within the half-open segment:  $\nu \in (0, 1/2]$ .*

REMARK 1. If  $\beta = 0$ , then equation (1) does not degenerate. However, as the degeneracy of equation (1) "increases" (that is, when parameter  $\beta$  increases from 0 to 1), the requirement for the smoothness of function  $f$  on the right side of equation (1) also increases accordingly.

According to Theorem 1 and the results of [2, 3], we have

**Theorem 2.** *Let  $\beta = 0$  and one of the following conditions is met*

$$f \in L^2(Q), \quad \partial_t f \in L^2(Q), \tag{4}$$

or

$$f \in L^2(Q), \quad (-\Delta)^{1/2} f \in L^2(Q). \tag{5}$$

*Then problem (1)–(3) (with a non-degenerate equation) is uniquely solvable in the space*

$$u(t) \in W_{2,1}^{2,2}(Q) \equiv \left\{ v(t) \mid v(t) \in L^2(0, T; W_2^2(\Omega) \cap \overset{\circ}{W}_2^{\frac{1}{2}}(\Omega)), \quad \partial_t^2 v(t) \in L^2(Q) \right\},$$

*and we have the corresponding a priori estimate*

$$\|u\|_{W_{2,1}^{2,2}(Q)}^2 \leq C \left[ \|f\|_{L^2(Q)}^2 + \|\partial_t f\|_{L^2(Q)}^2 \right], \tag{6}$$

or

$$\|u\|_{W_{2,1}^{2,2}(Q)}^2 \leq C \left[ \|f\|_{L^2(Q)}^2 + \|(-\Delta)^{1/2} f\|_{L^2(Q)}^2 \right]. \tag{7}$$

Note that, in part, the answers to these questions can be obtained using the theory developed in [2, 4]. We present one of the results following from ([2], chapter 5, Theorem 8.1, Remark 8.1 and Proposition 8.1).

**Theorem 3.** *Let the function  $f$  be given by the conditions*

$$f \in L^2(0, T; W_2^2(\Omega)), \quad \partial_t^3 f \in L^2(Q), \tag{8}$$

$$f(x, 0) = \partial_t f(x, 0) = \partial_t^2 f(x, 0) = 0. \tag{9}$$

Then the solution  $u(x, t)$  of problem (1)–(3) satisfies the inclusions

$$u \in W_{2,1}^{4,4}(Q) \Leftrightarrow u, \partial_t u \in L^2(0, T; W_2^4(\Omega) \cap \overset{\circ}{W}_2^1(\Omega)), \partial_t^4 u \in L^2(Q).$$

The following questions arise:

QUESTION 1. What can be said about the regularity of the solution of problem (1)–(3) for  $\beta = 1 \cup (1, 2) \cup 2$ ?

QUESTION 2. Is it possible to formulate the conditions of Theorem 1 in terms of the smoothness of the function  $f(x, t)$  with respect to the variable  $t$ , as it was done in Theorem 2 for the case  $\beta = 0$ ?

QUESTION 3. Is it possible to formulate conditions (8)–(9) of Theorem 3 in terms of the smoothness of the function  $f(x, t)$  with respect to the variable  $x$ ?

In our report, we provide answers to these and other related questions.

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**Keywords:** degenerate hyperbolic equation, sufficient condition of regular solvability.

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## Inequalities between Mixed Moduli of Smoothness

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Let  $L_{p_1 p_2}, 1 \leq p_i \leq \infty, i = 1, 2$  be the set of measurable functions of two variables  $f(x_1, x_2), 2\pi$ -periodic in each variable, for which  $\|f\|_{p_1 p_2} = \|\{ \|f\|_{p_1} \}\|_{p_2} < \infty$ , where

$$\|f\|_{p_i} = \left( \int_0^{2\pi} |f|^{p_i} dx_i \right)^{\frac{1}{p_i}}, \text{ if } 1 \leq p_i < \infty,$$

$$\|f\|_{p_i} = \sup_{0 \leq x_i \leq 2\pi} |f|, \text{ if } p_i = \infty.$$

Let  $L_{p_1 p_2}^0$  be the space of functions  $f \in L_{p_1 p_2}$  such that  $\int_0^{2\pi} f(x_1, x_2) dx_1 = 0$  for almost all  $x_2$  and  $\int_0^{2\pi} f(x_1, x_2) dx_2 = 0$  for almost all  $x_1$ .

For the function  $f \in L_{p_1 p_2}$ , we define the fractional differences of positive order  $\alpha_1$  and  $\alpha_2$  with steps  $h_1$  and  $h_2$  respectively, by variables  $x_1$  and  $x_2$  as follows:

$$\Delta_{h_1}^{\alpha_1}(f) = \sum_{\nu_1=0}^{\infty} (-1)^{\nu_1} \binom{\alpha_1}{\nu_1} f(x_1 + (\alpha_1 - \nu_1)h_1, x_2),$$

$$\Delta_{h_2}^{\alpha_2}(f) = \sum_{\nu_2=0}^{\infty} (-1)^{\nu_2} \binom{\alpha_2}{\nu_2} f(x_1, x_2 + (\alpha_2 - \nu_2)h_2),$$

where  $\binom{\alpha}{\nu} = 1$  for  $\nu = 0$ ,  $\binom{\alpha}{\nu} = \alpha$  for  $\nu = 1$ ,  $\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}$  for  $\nu \geq 2$ .

Denote ([3]) by  $\omega_{\alpha_1, \alpha_2}(f, \delta_1, \delta_2)_{p_1 p_2}$  the mixed modulus of smoothness of positive orders  $\alpha_1$  and  $\alpha_2$ , respectively, in the variables  $x_1$  and  $x_2$  of a function  $f \in L_{p_1 p_2}$ , that is,

$$\omega_{\alpha_1, \alpha_2}(f, \delta_1, \delta_2)_{p_1 p_2} = \sup_{|h_i| \leq \delta_i, i=1,2} \|\Delta_{h_1}^{\alpha_1}(\Delta_{h_2}^{\alpha_2}(f))\|_{p_1 p_2}.$$

The following  $(p, q)$ -inequality between moduli of smoothness in different metrics, nowadays called sharp Ulyanov type inequalities, is known (see [5]):

$$\omega_1(f, \delta)_q^{(1)} \ll \left( \int_0^{\delta} \left( t^{-\theta} \omega_1(f, t)_p^{(1)} \right)^{q^*} \frac{dt}{t} \right)^{\frac{1}{q^*}},$$

where  $1 \leq p < q \leq \infty, \theta = \frac{1}{p} - \frac{1}{q}$ . For the Lebesgue spaces, Ulyanov type inequalities in the one-dimensional case have been studied for the moduli of smoothness of any positive order by many authors, in particular, Ulyanov [5], V.I. Kolyada [1], Yu. Kolomoitsev, S. Tikhonov [2], B. Simonov [4].

**Theorem 1.** *Let  $f \in L_{p_1 p_2}^0$ , where  $1 < p_1 < q_1 < \infty$  or  $1 = p_1 < q_1 = \infty$  and  $1 < p_2 < q_2 < \infty$  or  $1 = p_2 < q_2 = \infty$ . Let for  $\alpha_i > 0, \delta_i \in (0, 1), i = 1, 2$ , we have*

$$\omega_{\alpha_1, \alpha_2}(f, \delta_1, \delta_2)_{q_1 q_2} \ll \left( \int_0^{\delta_2} \left( \int_0^{\delta_1} \left( t_1^{-\frac{1}{p_1} + \frac{1}{q_1}} t_2^{-\frac{1}{p_2} + \frac{1}{q_2}} \omega_{\alpha_1 + \frac{1}{p_1} - \frac{1}{q_1}, \alpha_2 + \frac{1}{p_2} - \frac{1}{q_2}}(f, t_1, t_2)_{p_1 p_2} \right)^{q_1^*} \frac{dt_1}{t_1} \right)^{\frac{q_2^*}{q_1^*}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2^*}}.$$

**Theorem 2.** *Let  $f \in L_{p_1 p_2}^0$ , where  $1 < p_1 < q_1 < \infty$  or  $1 = p_1 < q_1 = \infty$  and  $1 < p_2 < q_2 < \infty$  or  $1 = p_2 < q_2 = \infty$ . Let for  $\alpha_i > 0, \rho_i \geq 0, \delta_i \in (0, 1), i = 1, 2$ . we have*

$$\omega_{\alpha_1, \alpha_2}(f^{(\rho_1, \rho_2)}, \delta_1, \delta_2)_{q_1 q_2} \ll$$



$$\left( \int_0^{\delta_2} \left( \int_0^{\delta_1} \left( t_1^{-\rho_1 - \frac{1}{p_1} + \frac{1}{q_1}} t_2^{-\rho_2 - \frac{1}{p_2} + \frac{1}{q_2}} \omega_{\alpha_1 + \rho_1 + \frac{1}{p_1} - \frac{1}{q_1}, \alpha_2 + \rho_2 + \frac{1}{p_2} - \frac{1}{q_2}}(f, t_1, t_2)_{p_1 p_2} \right)^{q_1^*} \frac{dt_1}{t_1} \right)^{\frac{q_2^*}{q_1^*}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2^*}}.$$

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**Keywords:** Mixed moduli of smoothness of positive order, Ulyanov type inequality.

**2020 Mathematics Subject Classification:** 42A10, 41A25, 26A48.

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# On one Method for Solving a Boundary Value Problem for a Nonlinear Loaded Ordinary Differential Equation

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In the domain  $\bar{\Omega} = [0, \omega] \times [0, T]$  we considered semi-periodic boundary value problem for a semi-linear loaded hyperbolic equations:

$$\frac{\partial^2 u}{\partial x \partial t} = A(x, t) \frac{\partial u(x, t)}{\partial x} + A_0(x, t) \frac{\partial u(x, t)}{\partial x} \Big|_{x=x_0} + f\left(x, t, u(x, t), \frac{\partial u(x, t)}{\partial t}\right), \quad (1)$$

$$u(x, 0) = u(x, T), \quad x \in [0, \omega], \quad (2)$$

$$u(0, t) = \psi(t), \quad t \in [0, T], \quad (3)$$

where  $f : \bar{\Omega} \times R^2 \rightarrow R$ , continuous on  $\bar{\Omega}$ ,  $\psi(t)$  – continuously differentiable on  $[0, T]$  and satisfying the condition  $\psi(0) = \psi(T)$  functions.

Function  $u(x, t) \in C(\bar{\Omega})$ , having partial derivatives  $\frac{\partial u(x, t)}{\partial x} \in C(\bar{\Omega})$ ,  $\frac{\partial u(x_0, t)}{\partial x} \in C(\bar{\Omega})$ ,  $\frac{\partial u(x, t)}{\partial t} \in C(\bar{\Omega})$ ,  $\frac{\partial^2 u(x, t)}{\partial x \partial t} \in C(\bar{\Omega})$ , is called a classical solution of problem (1)-(3), if it satisfies equation (1) for all  $(x, t) \in \bar{\Omega}$  and the boundary conditions (2)-(3).

We introduce new unknown functions  $v(x, t) = \frac{\partial u(x, t)}{\partial x}$ ,  $w(x, t) = \frac{\partial u(x, t)}{\partial t}$ , and we must take into account that  $v(x_0, t) = \frac{\partial u}{\partial x} \Big|_{x=x_0}$ , and problem (1)-(3) is reduced to the following equivalent problem:

$$\frac{\partial v}{\partial t} = A(x, t)v(x, t) + A_0(x, t)v(x_0, t) + f(x, t, u(x, t), w(x, t)), \quad (4)$$

$$v(x, 0) = v(x, T), \quad x \in [0, \omega], \quad (5)$$

$$u(x, t) = \psi(t) + \int_0^x v(\xi, t) d\xi, \quad w(x, t) = \dot{\psi}(t) + \int_0^x w_t(\xi, t) d\xi \quad (6)$$

To find a solution to the boundary value problem (4)-(6), we use a modification of the Euler polyline method [1]. Using this method, we obtained conditions for the existence of boundary value problem (1)-(3). Boundary value problems for loaded differential equations have been studied by many authors [2-7].

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**Keywords:** loaded differential equation, impulse effect, parametrization method, algorithm.

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# A New Computational Approach for Solving Problem for Impulsive Differential Equations with Loadings

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We consider the following linear two-point boundary value problem for systems of essentially loaded differential equations with impulse effect:

$$\frac{dx}{dt} = A_0(t)x + \sum_{i=1}^m M_i(t) \lim_{t \rightarrow \theta_i+0} \dot{x}(t) + \sum_{i=1}^m A_i(t) \lim_{t \rightarrow \theta_i+0} x(t) + f(t), \quad t \in (0, T), \quad (1)$$

$$B_0x(0) + C_0x(T) = d, \quad d \in R^n, \quad x \in R^n, \quad (2)$$

$$B_i \lim_{t \rightarrow \theta_i-0} x(t) - C_i \lim_{t \rightarrow \theta_i+0} x(t) = \varphi_i, \quad \varphi_i \in R^n, \quad i = \overline{1, m}. \quad (3)$$

Here  $(n \times n)$ -matrices  $A_j(t)$ ,  $(j = \overline{0, m})$ ,  $M_i(t)$ ,  $(i = \overline{1, m})$ , and  $n$ -vector-function  $f(t)$  are piecewise continuous on  $[0, T]$  with possible discontinuities of the first kind at the points  $t = \theta_i$ ,  $(i = \overline{1, m})$ .  $B_j$  and  $C_j$ ,  $(j = \overline{0, m})$  are constant  $(n \times n)$  - matrices, and  $\varphi_i$ ,  $(i = \overline{1, m})$  are constant  $n$  vector functions,  $0 = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{m-1} < \theta_m < \theta_{m+1} = T$ .

A solution to problem (1) - (3) is a piecewise continuously differentiable vector function  $x(t)$  on  $[0, T]$  which satisfies the system of essentially loaded differential equations (1) on  $[0, T]$  except the points  $t = \theta_i$ ,  $(i = \overline{1, m})$ , the boundary condition (2), and conditions of impulse effects at the fixed time points (3).

Impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems [1]. Problems for impulsive differential equations with loadings and methods for finding their solutions are considered in [2].

Present work considers a two-point boundary value problem for the system of loaded ordinary differential equations with impulse effect (1) - (3). Loading points in differential equation are also the points of impulse effect. According to Dzhumabaev parametrization method [3] the interval is partitioned into parts, then the values of solution at the initial points of subintervals are introduced as additional parameters. By the matrices of the loaded summands, boundary condition, and conditions of impulse effect we construct a system of linear algebraic equations with respect to parameters. Coefficients and right-hand side of this system are determined by the solutions of Cauchy problems for linear ordinary differential equations. A numerical algorithm is offered for solving the problem (1) - (3).

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**Keywords:** loaded differential equation, impulse effect, parametrization method, algorithm.

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# Weighted Sobolev Type Inequalities and Identities

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Inspired by the work of Badiale-Tarantello [1], in this talk we discuss sharp remainder formulae for the cylindrical extensions of the improved Hardy inequalities with weights. For more general  $p$  we obtain cylindrical improved  $L^p$ -Hardy identities with weights for all real-valued functions  $f \in C_0^\infty(\mathbb{R}^n \setminus \{x' = 0\})$ , while in  $L^2$  case we have them for any complex-valued function  $f \in C_0^\infty(\mathbb{R}^n \setminus \{x' = 0\})$ . Moreover, we show cylindrical  $L^p$ -Hardy inequalities with weights for all complex-valued functions  $f \in C_0^\infty(\mathbb{R}^n \setminus \{x' = 0\})$ . As applications, we establish improved Caffarelli-Kohn-Nirenberg type inequalities with remainder terms. In addition, we also discuss the results in the setting of homogeneous Lie groups.

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**2020 Mathematics Subject Classification:** 35Q79, 35K05, 35K20

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## The Criterion of Minimality of the Mixed Type Gellerstedt Equation

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Minimal operators generated by overdetermined boundary conditions for differential equations are extremely important when describing regular boundary value problems for differential equations. In addition, for inverse problems of mathematical physics arising from applications, when determining unknown data, it is necessary to study problems with overdetermined boundary conditions, including the boundary conditions of minimal operators. Thus, the study of minimal differential operators is of both theoretical and applied interest. In this paper, a criterion for the regular solvability of the differential operator generated by the overdetermined Cauchy problem for the Gellerstedt equation is established. The proof is based on the Gellerstedt potential, the properties of solutions to the Goursat problem in the characteristic triangle, and the properties of special functions. It should be noted that differential operators of mixed type have numerous applications in transonic gas dynamics, the theory of infinitesimal surface bends, the instantaneous theory of shells with variable sign curvature, magnetodynamics and hydrodynamics.

Let  $\Omega \subset R^2$  be a domain, bounded by segment  $AB : y = 0, 0 < x < 1$ , and by characteristics  $AC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$  and  $BC : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$  of the Gellerstedt equation

$$Lu \equiv -(-y)^m u_{xx} + u_{yy} = f(x, y). \quad (1)$$

The overdetermined Cauchy problem is considered: to find a regular solution of (1) equation in the  $\Omega$  domain, satisfying the conditions:

$$u|_{y=0} = 0, \quad \left. \frac{\partial u}{\partial y} \right|_{y=0} = 0, \quad (2)$$

$$u|_{AC: x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0} = 0, \quad u|_{BC: x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1} = 0. \quad (3)$$

**Theorem 1.** *The minimal operator ((1), (2), (3) problem) is invertible in  $L_2(\Omega)$  if and only if the following conditions are met*

$$\int_0^\xi d\xi_1 \int_1^\xi \frac{(\eta_1 - \xi_1)^{2\beta}}{(\xi - \xi_1)^\beta (\eta_1 - \xi)^\beta} \cdot f_1(\xi_1, \eta_1) d\eta_1 = 0, \quad (4)$$

$$\int_0^\xi d\xi_1 \int_1^\xi \frac{\eta_1 - \xi_1}{(\xi - \xi_1)^{1-\beta} (\eta_1 - \xi)^{1-\beta}} \cdot f_1(\xi_1, \eta_1) d\eta_1 = 0, \quad (5)$$

where the function  $f_1$  is determined by a given function  $f(x, y)$  from (1). In this case, the solution of overdetermined Cauchy problem is representable by the formula:

$$u(\xi, \eta) = \int_0^\xi d\xi_1 \int_1^\eta R(\xi, \eta, \xi_1, \eta_1) \cdot f_1(\xi_1, \eta_1) d\eta_1, \quad (6)$$

where  $R(\xi, \eta, \xi_1, \eta_1)$  is the Riemann function of the Goursat problem.

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# On a Method for Constructing the Green's Function of the Dirichlet Problem and Regular Boundary Value Problem for the Laplace Equation

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Let  $\Omega$  – simple-connected domain with boundary  $\partial\Omega$ .

**Dirichlet problem.** find the solution for the Laplace equation in the domain  $\Omega$

$$-\Delta_x u = f(x), \quad x \in \Omega, \quad (1)$$

and satisfies the boundary condition

$$u|_{x \in \partial\Omega} = 0. \quad (2)$$

We call the function  $G(x, y)$  – the Green's function to the Dirichlet problem (1)-(2). The solution to this problem is represented as from

$$u(x) = \int_{\Omega} G(x, y) f(y) dy. \quad (3)$$

By using the fundamental solution of equation (1) finding the function  $G(x, y)$  reduces to a difficult second kind of Fredholm integral equation that does not have an effective solution. At this moment method of image is useful just for a ball and half plane.

Let  $D_S^{-1}$  define on the trace of simple layer potential and assume that  $(D_S^{-1}\nu)(x)$  – completely-continues and the self-adjoint operator

$$(D_S^{-1}\nu)(x) = \int_{\partial\Omega} \varepsilon(x, \xi) \nu(\xi) dS_{\xi}, \quad x \in \partial\Omega, \quad (4)$$

and denote  $D_S$  as inverse operator to  $D_S^{-1}$  that constructed by using spectral decomposition of the kernel  $\varepsilon_n(x - y)$ , where

$$\varepsilon_n(x, \xi) = \begin{cases} -\frac{1}{2} \ln |x - \xi|, & n = 2, \\ \frac{1}{(n-2)\sigma_n} |x - \xi|^{2-n}, & n \geq 3, \end{cases}$$

fundamental solution of the Laplace equation. The following theorem hold

**Theorem 1.** *Green's function of the Dirichlet problem (1)-(2) is represented in the form*

$$G(x, y) = \varepsilon_n(x, y) - \int_{\partial\Omega} \varepsilon_n(x, \xi) D_S \varepsilon_n(y, \xi) dS_{\xi}, \quad (x, y) \in \Omega. \quad (5)$$

**Theorem 2.** *Assume that  $q_0(\xi, y) \in W_2^\alpha(\partial\Omega) \cap L_2(\Omega)$ . Then for any  $f \in L_2(\Omega)$  exists unique solution of Laplace equation represented in the form*

$$u(x) = \int_{\Omega} \varepsilon(x, y) f(y) dy + \int_{\partial\Omega} \varepsilon(x, \xi) \int_{\Omega} q_0(\xi, y) f(y) dy dS_{\xi} \quad (6)$$

satisfying the inequality

$$\begin{aligned} \|L_Q^{-1} f\| &= \left\| \int_{\Omega} \varepsilon(x, y) f(y) dy + \int_{\partial\Omega} \varepsilon(x, \xi) \int_{\Omega} q_0(\xi, y) f(y) dy dS_{\xi} \right\| \\ &\leq c \|f\|_{L_2(\Omega)} + \|q_0\|_{W_2^\alpha(\partial\Omega) \cap L_2(\Omega)} \cdot \|f\|_{L_2(\Omega)}. \end{aligned}$$

**Theorem 3.** Assume that  $q_0(\xi, y) \in W_2^\alpha(\partial\Omega) \cap L_2(\Omega)$ ,  $-\Delta_y q_0(\xi, y) = 0$ , Then exists unique solution  $u \in W_2^2(\Omega)$  of equation  $-\Delta u = f$  satisfies boundary condition

$$N_Q[u]|_{x \in \partial\Omega} = -\frac{u(x)}{2} + \int_{\partial\Omega} \left( \frac{\partial \varepsilon(x, y)}{\partial n_y} + \int_{\partial\Omega} \varepsilon(x, \xi) \frac{\partial q_0(\xi, y)}{\partial n_y} dS_\xi \right) u(y) dS_\xi - \int_{\partial\Omega} \left[ \varepsilon(x, y) \frac{\partial u}{\partial n_y} + \int_{\partial\Omega} \varepsilon(x, \xi) q_0(\xi, y) \frac{\partial u}{\partial n_y} \right] dS_y = 0. \quad (7)$$

On the other hand, the function  $u \in W_2^2(\Omega)$  represented in the form (6) under the condition of theorem 3 satisfies boundary condition (7).

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- [2] Kal<sup>m</sup>menov T. S., Otelbaev M. oundary criterion for integral operators, *Doklady Mathematics*. “ Moscow : Pleiades Publishing, **93**:1 (2016), 58–61.

## Weighted Hardy-Type Inequalities for Monotone Functions

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Let  $I = (0, \infty)$ ,  $1 < p, q < \infty$  and  $p' = \frac{p}{p-1}$ . Suppose that  $v, u$  and  $v^{1-p'}$  are positive functions locally integrable on  $I$ .

We consider the following Hardy-type inequality

$$\left( \int_0^\infty u(x) \left| \int_0^x K(x,t)f(t)dt \right|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty v(x)|f(x)|^p dx \right)^{\frac{1}{p}}, \quad (1)$$

for all functions  $f \in L_{p,v}(I)$ , where  $L_{p,v}(I)$  is the weighted Lebesgue space such that  $\|f\|_{p,v} = \left( \int_0^\infty v(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty$ . Here

$$Kf(x) = \int_0^x K(x,t)f(t)dt, \quad x > 0, \quad (2)$$

is an integral operator with a non-negative kernel  $K(x,t)$ .

At present, there are many works devoted to Hardy-type inequalities with iterated operators. Motivated by important applications, all these generalizations of the Hardy inequality are studied not only on the cone of non-negative functions but also on the cone of monotone functions. This work discusses new Hardy-type inequalities for operators with kernels that satisfy less restrictive conditions than those considered earlier. The presented inequalities have already been characterized for non-negative functions. In this work, we continue this study but for monotone functions. To achieve the aim, we use the famous Sawyer duality principle, which gives an equivalence between the Hardy inequality for monotone functions and some inequality for non-negative functions.

# Boundedness of the Riesz operator in a Morrey type space

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Let  $\lambda \in \mathbb{R}$ ,  $0 < p, q \leq \infty$ . The Local Morry Space

$$LM_{p,q}^\lambda = \left\{ f : \left( \int_0^\infty \left( t^{-\lambda} \|f\|_{L_p(B_t(0))} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

where  $B_t(0) = \{y \in \mathbb{R}^n : |y| < t\}$  is the open ball centered at the point  $x = 0$  of radius  $t$ , were introduced by Guliyev-Burenkov [1]. In this paper, we introduce spaces that generalize local Morrey spaces. We study the Riesz operators, which are dedicated in [2,3]. O’Neil-type inequalities are obtained for convolution operators in these spaces. The methods from [4] are used.

**Theorem.** Let  $\mathbb{T}(u), \mathbb{T}(v)$  is a local partition of  $\mathbb{R}^n$  by points  $u, v$  respectively (see [4]). Let  $1 \leq p \leq \infty$ ,  $0 < \lambda < \frac{n}{p}$  and  $0 \leq \gamma \leq \frac{n}{p}$ ,  $0 < \alpha = \gamma - \lambda + \frac{n}{p}$ ,  $0 < \tau \leq \infty$ .

If  $f \in LM_{p,\infty}^\gamma(T(u-v))$  and  $g \in LM_{p',\tau}^{-\alpha}(T(v))$ , then  $f * g \in LM_{p,\tau}^\lambda(T(u))$  and

$$\|f * g\|_{LM_{p,\tau}^\lambda(T(u))} < c \|f\|_{LM_{p,\infty}^\gamma(T(u-v))} \|g\|_{LM_{p',\tau}^{-\alpha}(T(v))},$$

where the constant  $c$  depends only of  $n, \lambda, \alpha, p$ .

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**Keywords:** Morry space, convolution operator, Young-O’Neill inequality, Riesz potential.

**2020 Mathematics Subject Classification:** 47A30

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# Heat Polynomials Method for Solving Inverse Stefan Type Problems

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In a series of studies [1-4], we have developed the approximate heat polynomials method (HPM) for solving various inverse Stefan-type problems. The HPM, or polynomial Trefftz method [5], was initially described by Appell in 1892 in his paper [6], with further development in subsequent works [7-11].

The essence of this method lies in constructing an approximate solution that satisfies the heat conduction equation in the form of a linear combination of heat polynomials. The unknown coefficients are determined through the minimization of residuals, which can be achieved through various methods such as the variational method, the least squares method, or other techniques.

We have investigated the convergence and stability of the HPM through numerous numerical examples. A novel aspect of this work is the development of the collocation heat polynomials method for solving inverse Stefan-type problems.

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**Keywords:** inverse Stefan problems, approximate solution, heat polynomials method, moving boundary.

**2010 Mathematics Subject Classification:** 80A22, 80A23, 80M30, 35C11.

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## Nonlinear Stefan Problem with a Heat Source Arising in Closed Electrical Contacts

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This research introduces a mathematical framework that characterizes the heat dynamics occurring during the vaporization of metal in electrical contacts. The study establishes an arc erosion model for micro-asperities when in a liquid state, employing a one-phase Stefan problem approach. The investigation addresses two scenarios involving the impact of Joule heating on a generalized heat equation. By leveraging the similarity principle, the problems are simplified into ordinary differential equations for solution. The existence and uniqueness of these solutions are demonstrated using fixed point theory within a Banach space. Furthermore, a solution is presented for the specific scenario where thermal coefficients remain constant.

The mathematical formulation describing the behavior of the liquid phase involves a generalized heat equation that accommodates a bar with a varying cross-section:

$$c(\theta_1)\gamma(\theta_1)\frac{\partial\theta_1}{\partial t} = \frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\lambda(\theta_1)z^\nu\frac{\partial\theta_1}{\partial z}\right] + \rho(\theta_1)j^2, \quad z_1(t) < z < z_2(t), \quad 0 < t < t_a, \quad (1)$$

Within this context,  $\theta_1(z, t)$  signifies the temperature in the liquid phase. Parameters such as  $c(\theta_1)$ ,  $\gamma(\theta_1)$ ,  $\lambda(\theta_1)$ , and  $\rho(\theta_1)$  are specific heat, density, heat conductivity, and electrical resistivity, respectively, all of which depend on temperature and consequently introduce nonlinearity into the problem. The scaling symmetry of micro-asperity is captured by  $\nu > 1$ ,  $t_a$  represents the duration of arcing, and  $j$  symbolizes the current density. The boundary conditions are as follows:

$$\theta_1(z_1(t), t) = \theta_b, \quad (2)$$

$$\theta_1(z_2(t), t) = \theta_m. \quad (3)$$

Here,  $\theta_b$  denotes the boiling temperature, and  $\theta_m$  signifies the melting temperature. The position of the interface where the temperature reaches the melting point is identified as  $z = z_2(t)$  and can be determined by applying the Stefan condition:

$$-\lambda(\theta_1(z_2(t), t))\frac{\partial\theta_1(z_2(t), t)}{\partial z} = l_m\gamma_m\frac{dz_2}{dt}, \quad t > 0. \quad (4)$$

The latent heat and density at the melting temperature are represented as  $l_m$  and  $\gamma_m$ , respectively. The initial temperature field condition in this region is specified as:

$$\theta_1(z, 0) = \theta_m, \quad z_1(0) = z_2(0) = 0. \quad (5)$$

The Joule heating term in the equation (1) can be described in the form

$$\rho(\theta_1)j^2 = \rho(\theta_1)\frac{I_0^2\sin^2(\omega t)}{\pi^2 z^{2\nu}}$$

where

$$\sin^2(\omega t) \approx kt^{\nu-1}, \quad k = \frac{\sin^2(\omega t_a)}{t_a^{\nu-1}}.$$

We will examine the one more alternative problem that involves substituting condition (2) with a heat flux condition described by equation below:

$$-\lambda(\theta_1(z_1(t), t))\frac{\partial\theta_1(z_1(t), t)}{\partial z} = \frac{P_0e^{-z_0^2}}{2a\sqrt{\pi t}}. \quad (6)$$

In this context,  $P_0e^{-z_0^2}/(2a\sqrt{\pi t})$  represents a specified heat flux characterized by a positive power value  $P_0$ .

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**Keywords:** non-classical Stefan problem, generalized heat equation, Joule heat source, nonlinear thermal coefficients, fixed point theory.

**2020 Mathematics Subject Classification:** 35C11, 80A23, 80A22

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# Stein-Weiss-Adams Inequality on Morrey Space

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We establish Adams type Stein-Weiss inequality on global Morrey spaces on general homogeneous groups. Special properties of homogeneous norms and some boundedness results on global Morrey spaces play key roles in our proofs. As consequence, we obtain fractional Hardy, Hardy-Sobolev, Rellich and Gagliardo-Nirenberg inequalities on Morrey spaces on stratified groups. While the results are obtained in the setting of general homogeneous groups, they are new already for the Euclidean space  $\mathbb{R}^N$ . This work was jointly with M.A. Ragusa, M. Ruzhansky and D.Suragan and published in [1].

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**Keywords:** global Morrey space, Stein-Weiss inequality.

**2020 Mathematics Subject Classification:** 22E30, 43A80

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# The Algebraic Polynomial of Keller Polynomials

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In this talk, we consider Algebraic polynomials of Keller polynomials that were used in solving the Jacobian problem [1], [2], [3], [4]. It is known that the algebraic polynomial of two Keller polynomials in two variables is a heavy set [5], in the following article we show that the Keller polynomial for any minimal variable consists of only one point. It is clear that the basic method, in particular, analytic geometry, n variable n Keller polynomials in 2n affine systems gives an n-dimensional surface. The coordinates of any point on this surface are calculated on the n+1-dimensional plane and the n-dimensional surface. These coordinates are aligned with each other. This is the crucial stage of the proof. By deriving polynomials on the n+1 coordinates of this point, it is possible to use the Jacobian matrix of the ponomial representation defined by Keller polynomials.

**Keywords:** Jacobian problem, Jacobi matrix, polynomial representation, algebraic case, Keller polynomials.

**2020 Mathematics Subject Classification:** 13B25, 13B10

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## Equilibrium State in Economics and the Jacobian

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In Chapter 10 of the book (Oben, 1998), the state of equilibrium in economics is discussed in considerable detail, along with the mechanisms underlying the existence of an equilibrium state. A complex theorem proving the existence of a Walrasian equilibrium price has been demonstrated. The text specifically states: “The concept of economic equilibrium, to which we owe a debt to Walras, is by no means the only basis for our gratitude towards him. Since its first publication in 1859, Léon Walras, refuting the ideas of Proudhon, has shown that the mathematical method can be fruitful in economic theory.”

In 1974, in his book “Elements of Pure Economics,” Léon Walras proposed his conception of economic equilibrium as the solution to a system of nonlinear equations. The equality between the number of equations and the number of unknowns gave him enough confidence to assert the existence of a solution.

The book also introduces Walras’ Law and the Collective Walras’ Law. Utilizing these laws, it is proven that when the set of prices is convex and can be represented as the difference between a compact set and  $R_+^n$ , and when the set of consumption is convex, closed, and upper-bounded, then a Walrasian equilibrium price exists if the demand mapping satisfies the Collective Walras’ Law.

When considering polynomial mappings, all these conditions are replaced by the Jacobian. The Jacobian ensures uniqueness, but as Vitushkin’s example demonstrates, the Jacobian is not always a sufficient condition. An additional condition of contraction to a single point is also required. This paper elaborates on these positions in sufficient detail.

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**Keywords:** economic equilibrium, Walrasian equilibrium, Jacobian, nonlinear equations, demand mapping.

**2020 Mathematics Subject Classification:** 91B50, 91B52, 37N40

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# Solution of Initial-Boundary Value Problems for the Heat Equation with Discontinuous Coefficients

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The initial-boundary value problem for the heat equation with discontinuous coefficients

$$\frac{\partial u_i}{\partial t} = k_i^2 \frac{\partial^2 u_i}{\partial x^2}, \tag{1}$$

is considered in the domain  $\Omega = \cup \Omega_i$ ,  $\Omega_i = \{(x, t), l_{i-1} < x < l_i, 0 < t < T\}$  ( $i = 1, 2, 3$ ), with initial conditions

$$u(x, 0) = \varphi(x), \quad l_0 < x < l_3, \tag{2}$$

boundary conditions of the form

$$\begin{cases} \alpha_1 \frac{\partial u_1(l_0, t)}{\partial x} + \beta_1 u_1(l_0, t) = 0, \\ \alpha_2 \frac{\partial u_3(l_3, t)}{\partial x} + \beta_2 u_3(l_3, t) = 0, \end{cases} \quad 0 \leq t \leq T, \tag{3}$$

and pairing conditions

$$u(l_i - 0, t) = u(l_i + 0, t), \quad 0 \leq t \leq T \quad (i = 1, 2), \tag{4}$$

$$k_i \frac{\partial u_i(l_i - 0, t)}{\partial x} = k_{i+1} \frac{\partial u_{i+1}(l_i + 0, t)}{\partial x}, \quad 0 \leq t \leq T, \quad (i = 1, 2). \tag{5}$$

In the case without discontinuity, the spectral theory of these problems is constructed almost completely. In [1], the heat equation with a discontinuous coefficient was considered under Sturm-type boundary conditions (separated boundary conditions), eigenvalues and eigenfunctions were found, and various special cases were investigated.

In this paper, the spectral questions of problem (1)-(5) are investigated. The eigenvalues and eigenfunctions are found, and the existence and uniqueness theorem for the classical solution is proved.

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**Keywords:** Heat equation, discontinuous coefficients, eigenvalues, eigenfunctions, spectral theory.

**2020 Mathematics Subject Classification:** 35K05, 35K15

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## On the Fractional Neutral Levin-Nohel Integro-Differential Equations

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This work deals with the nonlinear neutral Levin-Nohel integro-differential equation with Caputo fractional derivative and variable delays. The Ulam-Hyers-Rassias stability, Ulam-Hyers stability, semi-Ulam-Hyers-Rassias stability are studied. The existence and uniqueness of solutions are established by using Krasnoselskii's fixed point theorem and contraction mapping principle.

**Keywords:** Levin-Nohel integro-differential equation, Caputo fractional derivative, existence of solution, uniqueness of solution, stability of solution, Krasnoselskii's fixed point theorem.

**2020 Mathematics Subject Classification:** 35Q79, 35K05, 35K20

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# **Lieb–Thirring Inequalities on Manifolds with Negative Curvature**

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We prove Lieb–Thirring inequalities on manifolds with negative constant curvature. The discrete spectrum appears below the continuous spectrum. In order to prove the main statement we use the result regarding Lieb–Thirring inequalities for Schrödinger operators with operator-valued potentials. As an application we obtain a Pólya type inequality with not a sharp constant

## On Hardy-Littlewood theorem

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This report is devoted to relation between integrability properties of functions and summability properties of their Fourier coefficients. In particular, we prove Hardy-Littlewood type theorem.

For functions  $f(x) \in L_1([0, 1])$  with Fourier series  $\sum_{k=-\infty}^{\infty} c_k e^{2\pi kx}$ , where  $\{c_k\}_{k=0}^{\infty}$  and  $\{c_k\}_{-\infty}^0$  are nonincreasing sequences the Hardy-Littlewood theorem holds i.e., there exist  $C_1, C_2 > 0$  such that

$$C_1 \|f\|_{L_p} \leq \left( \sum_{k=-\infty}^{\infty} (|k| + 1)^{p-2} c_k^p \right)^{\frac{1}{p}} \leq C_2 \|f\|_{L_p}. \quad (1)$$

Equivalence (1) has a lot of generalizations, see for instance, [1], [2], [3] and references therein. In this work, we get a new generalizations of Hardy-Littlewood's theorem. In particular, we obtain the equivalence (1) for functions with Fourier coefficients  $\{c_k\}_{k=-\infty}^{\infty}$  such that, for any  $k \geq 0$  and some  $C > 0$ , the following inequality

$$\sum_{[2^{k-1}] \leq |m| < [2^k]} |a_m - a_{m+1}| \leq C \sup_{r \in \mathbb{N}} \min(1, 2^{r-k}) \sup_{2^{r-1} \leq |m| < 2^r} \frac{1}{|m|} \left| \sum_{j=0}^m a_j \right|$$

holds, where  $[x]$  is an integer part of number  $x$ .

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**Keywords:** trigonometric series, general monotonicity, Hardy-Littlewood theorem.

**2020 Mathematics Subject Classification:** 42A16, 42A32

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## Mathematical Modeling of Fluid Filtration Processes with Consideration of Mass Transfer Processes

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The study is dedicated to approximate methods for solving the problem of non-isothermal filtration of immiscible fluids. The qualitative properties of solutions have been investigated. The obtained results were utilized in the development of digital technologies for oil and gas fields.

The problem formulation is considered through the Maskeet-Leverett temperature model.

The qualitative properties of solutions to problems (1) - (5) have been investigated, and computational algorithms have been developed for numerical implementation on a computer. Based on these algorithms, test cases with real data from a specific oil field in the western region of the Republic of Kazakhstan have been conducted.

$$\frac{\partial \Theta}{\partial t} = \text{div}(\lambda(x, s, \Theta)\nabla\Theta - \vec{v}\Theta), \tag{1}$$

$$m \frac{\partial s}{\partial t} = \text{div}(K a_0(a_1 \nabla\sigma - a_2 \nabla\Theta + \vec{f}_1) - b_1 \vec{v}) = \text{div} \vec{v}_1, \tag{2}$$

$$\text{div} \vec{v} = \text{div}(K(\nabla p + a_3 \nabla\Theta + \vec{f}_2)), \tag{3}$$

$$\sigma = \frac{s - s_*(\Theta)}{s^*(\Theta) - s_*(\Theta)}$$

on  $s_* \leq s(x, t) \leq s^*$ ;  $\sigma = 0$  on  $s < s_*(\Theta)$ ,  $\sigma = 1$  on  $s > s^*(\Theta)$  The last condition defines the function

$$\sigma = \Phi(s, \Theta)$$

at

$$s \in (0, 1),$$

where  $\Theta$  is equal to the temperature of the non-uniform liquid,  $\lambda$  is the thermal conductivity coefficient  $s = s_1$  is the saturation of the wetting phase

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

is the average filtration velocity of the mixture  $\vec{v}_i$  is the phase filtration velocities.

Additionally, in the considered model, the residual saturations are not constant, denoted as  $s_i^0 = s_i^0(\Theta) > s_i^{-0} = \text{const} > 0$ ,  $i=1,2$ . These specified properties lead to the following conditions for the saturation  $s_i^0$ ,  $i = 1, 2$   $s(x, t)$  (wetting phase):

$$0 \leq \text{const} = \vec{s}_* \leq s_*(\Theta) \leq s(x, t) \leq s^*(\Theta) \leq s^*(\Theta) \leq \vec{s}^* = \text{const} \leq 1$$

where  $\vec{s}_* = s_1^0(\Theta)$ ,  $\vec{s}^* = 1 - s_2^0(\Theta)$ .  $K = K(x, \Theta, \sigma)$  the tensor associated with the permeability of a medium.

$$a_0 = a_0(s), a_i(\sigma, \Theta), i = 1, 2, 3 \quad b_k = b_k(\sigma, \Theta), \quad \vec{f}_k = \vec{f}_k(x, \sigma, \Theta) \quad k = 1, 2$$

$$a_0(0) = a_0(1), b_1(0, \Theta) = 0; \inf a_1 > a_0 > 0$$

Let  $\Omega \in R^3$  be bounded region, the boundary  $\partial\Omega$  that is divided into several components depending on the type of boundary conditions:

$$(P, S, \Theta) = (P_0, S_0, \Theta_0), (x, t) \in \Sigma^1 = \Gamma^1 x(0, T)$$



$$\begin{aligned}\vec{v}_i \vec{n} &= b_i \psi, \quad i = 1, 2; \Theta = \Theta_0(x, t), \quad (x, t) \in \Sigma^2 = \Gamma^2 x(0, T) \\ \vec{v}_i \vec{n} &= b_i \psi, \quad i = 1, 2; \\ \lambda \frac{\partial \Theta}{\partial n} &= \beta (\Theta_0 - \Theta), \quad (x, t) \in \Sigma^3 = \Gamma^3 x(0, T)\end{aligned}$$

To the boundary conditions (4), it is necessary to add initial conditions:

$$(s, \Theta)_{t=0} = (s_0, \Theta_0)(x, 0), \quad x \in \Omega$$

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**2020 Mathematics Subject Classification:** 35Q79, 35K05, 35K20

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# Initial-Boundary Value Problem for the Heat Equation under Antiperiodic Boundary Conditions in the Absence of Agreement Between the Initial and Boundary Data

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In the report, the initial-boundary value problem for the heat equation under antiperiodic boundary conditions in the absence of agreement between the initial and boundary data is considered.

It is well known that when studying boundary value problems for parabolic equations in Hölder space, it is necessary to require the fulfillment of conditions for the compatibility of boundary and initial data at the boundary of the domain at  $t = 0$ . These conditions ensure the continuity of the solution and its derivatives, as well as the boundedness of the Hölder constants of higher derivatives inside the domain. Consistency conditions are functional identities that connect these functions on the boundary of the domain at the initial moment.

**Problem P.** Find in  $\Omega = \{(x, t) : 0 < x < l, 0 < t < T\}$  a solution  $u(x, t)$  of the heat equation

$$u_t - k^2 u_{xx} = f(x, t), \tag{1}$$

satisfying the initial condition

$$u(x, 0) = \varphi(x), 0 \leq x \leq l, \tag{2}$$

and antiperiodic boundary conditions

$$\begin{cases} u_x(0, t) + u_x(l, t) = 0, \\ u(0, t) + u(l, t) = 0. \end{cases} \tag{3}$$

It is well known that for the existence of a classical solution, the matching conditions must be satisfied. For example, the zero and first order matching conditions for problem (1)-(3) are

$$A_0 \equiv \varphi(0) + \varphi(l) = 0, \quad A_1 \equiv \varphi'(0) + \varphi'(l) = 0.$$

The second-order matching condition arises when we consider solutions of the problem from the class  $u \in C_{x,t}^{2,1}(\bar{\Omega})$ . For functions from such a class, we can pass to the limit in equation (1) at  $t \rightarrow 0$  at  $x = 0$  and  $x = l$ . Then we get

$$A_2 \equiv k^2[\varphi''(0) + \varphi''(l)] - [f(0, 0) + f(l, 0)] = 0.$$

For a problem with Dirichlet boundary conditions, solutions with a mismatch between the boundary and initial data were studied in [1].

In this report, we consider the problem (1)-(3) with nonlocal boundary conditions. Cases where the matching conditions of the zero, first and second orders are not met are considered in the report.

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**Keywords:** heat equation, initial-boundary value problem, nonlocal boundary conditions, matching conditions.

**2020 Mathematics Subject Classification:** 35K05, 335K15

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## On a Semi-Minimal Degenerate Hyperbolic Operator

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Let  $\Omega \subset R^n$  be bounded simply-connected domain with smooth boundary  $\partial\Omega$ , and denote  $D = (0, T) \cap \Omega$ .

Let us consider the following equation on the domain  $D$

$$Lu = \frac{\partial}{\partial t} \left( K(t) \frac{\partial}{\partial t} u \right) - \Delta_x u = f(x, t), \tag{1}$$

where  $K(t) = t^\beta (1-t)^\beta K_1(t)$ ,  $K_1(t) \in C^2[0, T]$ .

Denote  $L_0$  as the closure of the differential operator (1) in  $L_2(D)$  on a subset of functions  $u \in \overset{\circ}{W}_2^2(0, T) \cap C^2(\overline{\Omega})$ , which satisfies the lateral potential boundary condition

$$-\frac{u(x, t)}{2} - \int_{\partial\Omega} \left( \varepsilon_n(x, \xi) \frac{\partial u(\xi, t)}{\partial n_\xi} - \frac{\partial \varepsilon_n(x, \xi)}{\partial n_\xi} u(\xi, t) \right) dS_\xi = 0, \quad x \in \partial\Omega, \quad t \in [0, T]. \tag{2}$$

where  $\varepsilon(x, \xi)$  is the fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x, \xi) = \delta(x - \xi), \quad x, \xi \in \Omega. \tag{3}$$

By using spectral decomposition to the self-adjoint problem

$$-\Delta_x e_m(x) = \lambda_m e_m(x), \quad x \in \Omega, \tag{4}$$

$$-\frac{e_m(x)}{2} - \int_{\partial\Omega} \left( \varepsilon_n(x, \xi) \frac{\partial e_m(\xi)}{\partial n_\xi} - \frac{\partial \varepsilon_n(x, \xi)}{\partial n_\xi} e_m(\xi) \right) dS_\xi = 0, \quad x \in \partial\Omega, \tag{5}$$

have found reversibility conditions for the semi-minimal operator  $L_0^{-1}$ .

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## Methods of Stated and Revealed Preferences: A Case Study in Healthcare Services

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Responsibility for personal health is at the same time immediate and challenging for rational decision-making. Immediate because health problems are intimately felt by everyone, and no one wants to live with pain. Challenging because many of health problems are potential, unobservable to the individual, and hence are likely to be neglected as non-existent or impossible<sup>TM</sup>. Taking them into proper account requires time consistency, forward-looking behaviour and objective assessment of probabilities of future states “yet exactly these features are known to be problematic in light of behavioural research (Kahneman et al., 1992; 2000).

We explore rationality of forward-looking behaviour in an example of drug insurance program which is currently under consideration in several countries, including, in various forms, Australia, Canada, Norway, Switzerland and others. As empirical base for our study we take the case of Kazakhstan whose government considers co-funding of drug insurance by the citizens and the government should the citizens need it. Specific scenario that we consider is that a participating citizen contributes an amount of her disposable income, which is automatically doubled by the government. This doubled amount is reserved on citizen<sup>TM</sup>s personal account, and can be used within the next year to procure with needed drugs free of charge. (We do not consider here details of this procurement mechanism, such as its constraining to particular pharmacies, conditioning by prescription, availability of drugs, or possibility of their sharing with other people.). Rational individual should apparently purchase drug insurance for the amount needed to cover their expected needs, provided the benefits of drugs exceed the costs of their out-of-pocket purchases.

Do real people always behave in that way? Abundant research in behavioural economics suggests they might not. People do not properly check their own health state (Thornton, 2008), even if these are pertinent to potentially life-threatening conditions (Haisley et al., 2012), ignore prescriptions of covered drugs (Kling et al., 2012), which calls for special efforts to nudge them to do so (Keller et al., 2011; Kimmel et al., 2012). Further, there is evidence suggesting that ignorance of relevant information indeed causes welfare losses (Liebman and Luttmer, 2015).

Baseline model. Following the traditional economics approach to health decisions (Cardon and Hendel, 2001; Finkelstein et al., 2019), we consider a rational von Neumann-Morgenstern expected utility maximizer who may turn out to be in two health states bad health state  $h_B$  or in good one,  $h_G$ . We treat disease as deterioration of health to  $h_B < h_G$ , and normalize  $h_G$  to 0. Suppose the subjective probabilities of states  $h_B$  and  $h_G$  are  $p$  and  $(1-p)$ , respectively. Rational decision maker should then name the price of drug insurance policy  $z$  knowing it will be doubled and used towards her recovery in case of bad state. Utility maximization of a risk-neutral individual requires

$$(1-p)c + p(c - \phi(h_B)) = (1-p)(c-z) + p[(c-z) - \phi(h_B + g(2z))]$$

where  $c$  is consumption of ordinary (health-unrelated) goods,  $c-z$  “consumption short of cost of health insurance,  $g(\cdot)$  is function of health recovery with drug expenditure  $2z$ , and  $\phi$  is marginal utility of health. This is a straightforward balancing model, but several assumptions are implicit in it.

1. Utility of conventional consumption  $c$  is linear in its monetary volume (money-metric utility).
2. Probabilities of states are known to the decision maker.

3. Marginal utility of health  $\phi$  measured in utility units is objectively known and additive separable from utility from conventional consumption.
4. Deterioration of health  $hB$  is known, as is the cost of its recovery.
5. Drug insurance is the only ex ante mean of recovery in case of illness.
6. Decision maker opts for full recovery of health after the disease.
7. Out-of pocket coverage of drug expenses is not allowed.

The above story closely mimics Finkelstein et al., 2019, except that their story is developed for the US medical insurance (Medicaid), which they solve for optimal behaviour, equation (2). Our addition to this model is the possibility of boundedly rational mis-optimization: instead of What we do over that is to consider the possibility of suboptimal behaviour: instead of investing in drug insurance to decrease one's total medical expenses one may think of suboptimal insurance resulting in lower  $m$  and lower-than-optimal health  $h()$ .

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**Keywords:** healthcare economics, rational decision-making, behavioral economics, drug insurance, bounded rationality.

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# Numerical Modeling Nonlinear Processes Heat Conductivity in Componential Media

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Consider in the domain  $Q = \{(t, x) : t > 0, x \in R^N\}$  a parabolic system of two quasilinear equations of non-divergence

$$(19) \quad \begin{aligned} \frac{\partial u}{\partial t} &= u^{\alpha_1} \nabla \left( |x|^n u^{m_1-1} |\nabla u^k|^{k-2} \nabla u^{l_1} \right) + \varepsilon u^{p_1} v^{q_1}, \\ \frac{\partial v}{\partial t} &= v^{\alpha_2} \nabla \left( |x|^n v^{m_2-1} |\nabla v^k|^{k-2} \nabla v^{l_2} \right) + \varepsilon u^{p_2} v^{q_2}, \end{aligned}$$

$$(20) \quad u|_{t=0} = u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) \geq 0, \quad \forall x \in R^N$$

where  $\varepsilon = \pm 1$ ,  $p_i, q_i \geq 1$ ,  $p \geq 2$ ,  $k, m_i > 1$ ,  $\alpha_i$  ( $i = 1, 2$ ), are positive real numbers, and  $u_0(x) \geq 0, v_0(x) \geq 0$  are a non-trivial, non-negative, bounded and sufficiently smooth function,  $p \geq n$ ,  $u = u(t, x) \geq 0, v = v(t, x) \geq 0$  are the desired solution. The variable  $x$  is parameter heat conductivity and parameter  $n$  identifies the variable density of the nonlinear medium. Non-linear equations and systems of equations in a non-divergent form of the form (19) are often used to describe various physical phenomena, such as the diffusion process for biological species, resistive diffusion phenomena in non-forced magnetic fields, infection propagation processes, and so on, (see [1 -3] and the literature cited there).

In this work, based on self-similar analysis, an estimate of solutions and a front for problem (19)-(20) is obtained. A self-similar system of equations has been constructed for non-linear degenerate system of equations of a parabolic type of a non-divergent type (1) with a variable density under the influence of nonlinear source or absorption. The value of the numerical parameters and the asymptotic of solutions has been obtained depending on the value of the numerical parameters. The value of the numerical parameters and the asymptotic of solutions has been obtained depending on the value of the numerical parameters.

**Keywords:** non-divergence, nonlinear, self-similar system, parabolic type, variable density, fast and slow diffusion, blow up **2020 Mathematics Subject Classification:** 35C06, 35D30, 517.958

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# Weighted Estimates for one Class of Quasilinear Integral Operators with Kernels

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Let  $I = (0, \infty)$ ,  $1 < p, r, q < \infty$  and  $p' = \frac{p}{p-1}$ . Let  $u, v$  and  $w$  be positive functions locally summable on  $I$  such that  $u \in L_q^{loc}(I)$  and  $v^{-1} \in L_{p'}^{loc}(I)$ . Let

$$\|f\|_{p,v} = \left( \int_0^\infty |v(t)f(t)|^p dt \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty.$$

We consider the weighted inequality

$$\|Rf\|_{q,u} \leq C\|f\|_{p,v}, \quad f \geq 0, \tag{1}$$

where  $R$  is one of the following operators

$$Tf(x) = \left( \int_0^x G(x,t)w(t) \left( \int_t^\infty K(s,t)f(s)ds \right)^r dt \right)^{\frac{1}{r}}, \tag{2}$$

$$Sf(x) = \left( \int_0^x G(x,t)w(t) \left( \int_0^t K(t,s)f(s)ds \right)^r dt \right)^{\frac{1}{r}}, \tag{3}$$

$$T^-f(x) = \left( \int_x^\infty G(t,x)w(t) \left( \int_0^t K(t,s)f(s)ds \right)^r dt \right)^{\frac{1}{r}}, \tag{4}$$

$$S^-f(x) = \left( \int_x^\infty G(t,x)w(t) \left( \int_t^\infty K(s,t)f(s)ds \right)^r dt \right)^{\frac{1}{r}}. \tag{5}$$

The inequality (1) for the operators (2)–(5) with  $G(\cdot, \cdot) \equiv K(\cdot, \cdot) \equiv 1$  has been studied, for example, in the works [1] and [2]. The case, when  $G(\cdot, \cdot) \equiv 1$  and  $K(\cdot, \cdot)$  satisfies the condition  $\mathcal{O}$  stating that  $K(x, s) \geq 0$  for  $x \geq s > 0$  and  $K(x, s) \approx K(x, t) + K(t, s)$  for  $x \geq t \geq s > 0$ , has been considered, for example, in [3].

A more general case, when  $G(\cdot, \cdot)$  and  $K(\cdot, \cdot)$  satisfy the condition  $\mathcal{O}$ , has been investigated, for example, in [4] and [5].

Here we present the results for the inequality (1) when  $R \equiv T$  and  $R \equiv S$ . Similar results are obtained for  $R \equiv T^-$  and  $R \equiv S^-$ .

We assume that the functions  $K(\cdot, \cdot)$  and  $G(\cdot, \cdot)$  belong to the classes  $\mathcal{O}_n^\pm, n \geq 1$ , and  $\mathcal{O}_m^\pm, m \geq 1$ , respectively. The classes  $\mathcal{O}_n^\pm, n \geq 1$ , were introduced in [6]. These classes are wider than the class  $\mathcal{O}$ .

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**Keywords:** Integral operator, Hardy-type inequality, Weight function, Kernel.

**2020 Mathematics Subject Classification:** 26D10, 26D15

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# Navier-Stokes-Voigt Equations Governing Density-Dependent Flows with Vacuum

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Let us consider the following initial-boundary value problem for the Navier-Stokes-Voigt system that governs density-dependent flows (nonhomogeneous flows) of incompressible fluids with elastic properties,

$$\operatorname{div} u = 0 \quad \text{in } Q_T, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) = \rho f - \nabla p + \mu \Delta u + \kappa \Delta u_t \quad \text{in } Q_T, \tag{2}$$

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad \rho \geq 0 \quad \text{in } Q_T, \tag{3}$$

$$\rho u = \rho_0 u_0, \quad \rho = \rho_0 \quad \text{in } \{0\} \times \Omega, \tag{4}$$

$$u = 0 \quad \text{on } \Gamma_T. \tag{5}$$

Here,  $\Omega \subset \mathbb{R}^d$  is a bounded domain,  $Q_T = (0, T) \times \Omega, 0 < T < \infty$  is a cylinder with lateral  $\Gamma_T = (0, T) \times \partial\Omega$ . The unknowns of the problem are  $u, \rho$  and  $p$ , while  $f, u_0$  and  $\rho_0$  are given data. For now, we consider a general space dimension  $d \geq 2$ , though the real-world applications correspond to the cases  $d = 2, 3$ . In these cases,  $u$  denotes the velocity field,  $\rho$  accounts for the density,  $p$  is the pressure,  $f$  stands for the external forces field, while  $\mu$  and  $\kappa$  are positive constants to the dynamic viscosity and to the relaxation time. We are interested in the case of the initial data  $u_0$  and  $\rho_0$  satisfying

$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega,$$

$$0 \leq \rho_0 \leq M < \infty \quad \text{in } \Omega,$$

for some positive constant  $M$ . The case when  $0 < m \leq \rho \leq M < \infty$  was considered in [1].

In this talk, we prove the existence and uniqueness of strong solution above posed problem (1)-(5).

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**Keywords:** Navier-Stokes-Voigt model, incompressible fluids with non-constant density, nonhomogeneous and incompressible fluids, existence.

**2020 Mathematics Subject Classification:** 35Q35, 76D03, 76A10

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# Initial Boundary Value Problems for the Wave Equation with Not Strongly Regular Boundary Conditions

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Initial-boundary value problems for linear equations of hyperbolic type are a fairly well-developed part of the theory of partial differential equations (see, for example, [1-3]). And one of the most developed methods for solving them is the Fourier method, which is also called the method of separation of variables or the method of expansion in terms of eigenfunctions. This method has been well developed for the case of self-adjoint boundary conditions in a spatial variable. However, for the case of non-self-conjugate boundary conditions the problem still remains open.

In this talk considers initial boundary value problems for the one-dimensional wave equation

$$u_{tt}(x, t) - u_{xx}(x, t) + q(x)u(x, t) = f(x, t), \quad (x, t) \in \Omega, \quad (1)$$

in the domain  $\Omega = \{(x, t) : 0 < x < 1, 0 < t < T\}$ , with nonlocal boundary conditions of general form

$$U_j(u) = a_{j1}u_x(0, t) + a_{j2}u_x(1, t) + a_{j3}u(0, t) + a_{j4}u(1, t) = 0, \quad j = 1, 2. \quad (2)$$

Additionally, standard initial conditions are specified

$$u(x, 0) = \tau(x), \quad u_t(x, 0) = \nu(x), \quad 0 \leq x \leq 1. \quad (3)$$

Application of the Fourier method (method of separation of variables) leads to the following spectral problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad 0 < x < 1, \quad (4)$$

$$U_j(y) = a_{j1}y'(0) + a_{j2}y'(1) + a_{j3}y(0) + a_{j4}y(1) = 0, \quad j = 1, 2. \quad (5)$$

It is well known that if conditions (5) are strongly regular, then the system of root vectors of problem (4)-(5) forms a Riesz basis in  $L_2(0, 1)$ . And the Fourier method can be implemented to solve problem (1)-(3). However, when the boundary conditions (5) are not strongly regular, the system of root vectors of problem (4)-(5) may not form an unconditional basis. And this does not make it possible to use the Fourier method. In the case when the boundary conditions (5) are irregular, the system of root vectors of problem (4)-(5) does not form an unconditional basis. Thus, for the application of the Fourier method, the case when the boundary conditions (5) are not strongly regular remains unfounded.

This talk considers just such a case. An algorithm has been constructed to prove the correctness (in the classical and generalized senses) of the initial boundary value problem (1)-(3) for the case when the boundary conditions (5) are not strongly regular. This method can be applied regardless of whether the system of root vectors of problem (4)-(5) forms an unconditional basis in  $L_2(0, 1)$  or not.

The technique is based on the method for solving heat conduction problems with not strongly regular boundary conditions [4].

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**Keywords:** wave equation, nonlocal boundary condition, not strongly regular boundary conditions, Riesz basis, Fourier method.

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## Improved Hardy-Sobolev Inequalities

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In this talk, we obtain a new version of the Hardy-Sobolev inequality in the integral form which covers the recent inequality derived in [1] and improves the results established in [2]. It gives new results in one dimension. Moreover, we analyse radial and non-radial versions of the considered multidimensional sharp Hardy-Sobolev inequality and as a consequence we establish an improved version of the Heisenberg-Pauli-Weyl uncertainty principle.

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**Keywords:** Hardy-Sobolev inequality, Sharp Constant, Symmetric rearrangements, Uncertainty principle.

**2010 Mathematics Subject Classification:** 26D10, 35A23, 46E35

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# On the Third-Order Differential Equation with Unbounded Coefficients

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We study the solvability and maximal regularity questions for the singular third-order differential equation with unbounded coefficients and some applications. In contrast to the cases studied earlier, the leading and intermediate coefficients of above equation can grow independently.

We obtain sufficient coefficient conditions for the correctness of the equation and compactness of the inverse to the corresponding third-order differential operator. The maximal regularity estimate for a generalized solution is proved. Using these results we have obtained upper and lower estimates for the number of Kolmogorov  $k$ -diameters of one set associated with solutions of the linear Korteweg-de Vries equation. For specific coefficients, we have shown that, two-sided estimates of the Kolmogorov  $k$ -diameters themselves are derived from the maximal regularity inequality.

Methods of the theory of closed operators, embedding and compactness theorems for weighted Sobolev spaces, as well as some integral inequalities of Hardy type with weights are used in the work.

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**Keywords:** singular differential equation, generalized solution, correctness, coercive estimate, resolvent, compactness.

**2020 Mathematics Subject Classification:** 34B40, 34D05, 47A58

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# On Logarithmically Submajorization for $\tau$ - Measurable Operators

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We denote by  $\mathcal{M}$  a semi-finite von Neumann algebra with a faithful normal finite trace  $\tau$  and by  $L_0(\mathcal{M})$  the set of all  $\tau$ -measurable operators associated with  $(\mathcal{M}, \tau)$ . For  $x \in L_0(\mathcal{M})$ , the distribution function  $\lambda \cdot(x)$  of  $x$  is defined by  $\lambda_t(x) = \tau(e_{(t, \infty)}(|x|))$  for  $t > 0$ , where  $e_{(t, \infty)}(|x|)$  is the spectral projection of  $|x|$  in the interval  $(t, \infty)$ , and the generalized singular numbers  $\mu \cdot(x)$  of  $x$  by

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\} \quad \text{for } t > 0.$$

Let

$$L_{\log_+}(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \log_+ |x| \in L_1(\mathcal{M}) + \mathcal{M}\},$$

where  $\log_+ t = \{\log t, 0\}$ ,  $t > 0$ . If  $x, y \in L_{\log_+}(\mathcal{M})$  such that

$$\int_0^t \log \mu_s(x) ds \leq \int_0^t \log \mu_s(y) ds, \quad t > 0,$$

$x$  is said to be logarithmically submajorized by  $y$ , denoted by  $x \preccurlyeq_{\log} y$ .

**Theorem 1.** *The following statements are equivalent:*

(i) *If  $x, y \in L_{\log_+}(\mathcal{M})$  are self-adjoint operators such that  $\pm y \leq x$ , then  $y \preccurlyeq_{\log} x$ .*

(ii) *If  $a, b \in \mathcal{M}$ ,  $x, y \in L_{\log_+}(\mathcal{M})$  and  $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ , then*

$$a^* z b + b^* z^* a \preccurlyeq_{\log} a^* x a + b^* y b.$$

(iii) *If  $x, y, z \in L_{\log_+}(\mathcal{M})$  and  $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ , then  $z^* + z \preccurlyeq_{\log} x + y$ .*

(iv) *If  $x, y \in L_{\log_+}(\mathcal{M})$  are positive operators, then  $x - y \preccurlyeq_{\log} x + y$ .*

(v) *If  $x, y, z \in L_{\log_+}(\mathcal{M})$  and  $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ , then  $z^* \oplus z \preccurlyeq_{\log} x \oplus y$ .*

(vi) *If  $x, y \in L_{\log_+}(\mathcal{M})$  are normal operators and  $z \in L_{\log_+}(\mathcal{M})$  is positive operator, then for any contraction  $a \in \mathcal{M}$ ,*

$$|za(x+y)a^*z| \preccurlyeq_{\log} za(|x|+|y|)a^*z.$$

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**Keywords:** generalized singular value,  $\tau$ -measurable operator, semi-finite von Neumann algebra.

**2020 Mathematics Subject Classification:** 46L52, 47L05

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## Finite-Time/Fixed-Time Synchronization of Memristive Shunting Inhibitory Cellular Neural Networks via Sliding Mode Control

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This abstract proposes a novel time-dependent gain parameter-based sliding mode controller (SMC) to realize the finite-time/fixed-time synchronization of memristive shunting inhibitory memristive neural networks (Mem-SICNNs) having time-varying delays. In this regard, a new terminal sliding mode surface is designed and its reachability is analysed. According to synchronization error analysis, the stability property of the desired error system is reached within finite-time/fixed-time range by proposing a unique time-dependent gain parameter-based SMC and choosing the appropriate Lyapunov functionals. Finally, a numerical example is approached by software simulation and manual calculation to estimate the settling-time of the finite-time/fixed-time synchronization criteria of the proposed Mem-SICNNs model.

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**Keywords:** finite-time/fixed-time synchronization; memristor; shunting inhibitory cellular neural networks; sliding mode control.



# Generalized Stankovich Transform and Distributed Order Evolutionary Equations

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The report discusses an integral transform connecting the operators  $\frac{d}{dx}$  and  $\mathcal{D}_x^{[\mu]}$ , where

$$\mathcal{D}_x^{[\mu]} f(x) = \int D_x^t f(x) \mu(dt)$$

is a distributed order differentiation operator. Here  $D_x^t$  denotes a fractional derivative of order  $t$  with respect to  $x$ ,  $\mu$  is a non-negative Lebesgue-Stieltjes measure. It is assumed that  $\text{supp } \mu \in [0, 1)$  and  $\text{sup } \text{supp } \mu > 0$ . In the case when  $\mu$  is concentrated at a point, the operator  $\mathcal{D}_x^{[\mu]}$  coincides (up to a constant factor) with the fractional differentiation operator, and the corresponding transform turns into the Stankovich transform [1, 2].

As an application of the considered transform, we construct solutions of initial problems for the distributed order evolutionary equation of the form

$$\mathcal{D}_x^{[\mu]} u(x) = Lu(x) + f(x), \quad \lim_{x \rightarrow 0} \mathcal{D}_x^{[\mu_1]} u(x) = a,$$

in terms of solutions of the problem

$$v'(x) = Lv(x) + g(x), \quad v(x) = a.$$

Here  $L$  is a linear operator that does not depend on  $x$  (it is assumed that  $u(x)$ ,  $f(x)$  and  $a$ , as well as  $L$ , can depend on other variables, i.e. can be elements of some function space, for each fixed  $x$ ), and  $\mu_1$  is the shift of the measure  $\mu$  by 1.

**Keywords:** fractional derivative, distributed order differentiation operator, evolutionary equation, Stankovich transform.

**2020 Mathematics Subject Classification:** 26A33, 35A22, 35R11, 34A08

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# Solvability of Boundary Value Problems for a Involutory Second-Order Differential Equation

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We study the existence of a solution to the differential equation

$$y''(x) + \alpha y''(-x) = F(x, y(x), y(-x)), \quad x \in (-1, 1), \tag{1}$$

with boundary conditions

$$y(-1) = y_1, \quad y(1) = y_2, \tag{2}$$

where  $F : [0, 1] \times R^2 \rightarrow R$  is a given function,  $\alpha \neq \pm 1$ . The boundary value problem (1), (2) is equivalent to the integral equation

$$y(x) = \frac{1}{2}(y_2 + y_1) + \frac{1}{2}(y_2 - y_1)x + \int_{-1}^1 G(x, t) F(t, y(t), y(-t)) dt,$$

where  $G(x, t)$  is Green's function of the homogeneous boundary value problem (1), (2) in the case  $y_1 = y_2 = 0$ .

**Theorem 1.** *Let  $\alpha \neq \pm 1$ . Let the function  $F(x, \varsigma, \xi)$  be continuous and satisfy the Lipschitz condition  $|F(x, \varsigma, \xi) - F(x, \tilde{\varsigma}, \tilde{\xi})| \leq l_1 |\varsigma - \tilde{\varsigma}| + l_2 |\xi - \tilde{\xi}|$  for any  $(x, \varsigma, \xi), (x, \tilde{\varsigma}, \tilde{\xi}) \in \Omega$ , with some positive numbers  $l_1, l_2$  such that  $\frac{9(l_1+l_2)}{16|1-\alpha|} < 1$ . Then the boundary value problem (1), (2) has a unique solution.*

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**Keywords:** second-order differential equation with involution; Green's function; non-linear equation; boundary value problem; Schauder fixed point theorem.

**2020 Mathematics Subject Classification:** 34A34, 34B27, 34K10

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# Inverse Problem of Determining Time-Dependent Source Term in the Time-Fractional Heat Equation

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Let  $\mathcal{H}$  be a separable Hilbert space and let  $\mathcal{L}$  be operator with a discrete spectrum on  $\mathcal{H}$ . For

$$\begin{cases} \mathcal{D}_t^\alpha u(t) + a(t)\mathcal{L}u(t) = r(t)g \text{ in } \mathcal{H}, 0 < t \leq T, \\ u(0) = h \text{ in } \mathcal{H}, \end{cases} \quad (1)$$

we study

**Inverse source problem.** Given  $a(t)$ ,  $g$  and  $h$ , find a pair of functions  $(r, u)$  satisfying the problem (1) and the additional condition

$$F[u(t)] = E(t), \quad t \in [0, T], \quad (2)$$

where  $F$  is the linear bounded functional.

As for this kind of inverse problems for parabolic equation, see [1,2] for example. In this work, we prove the well-posedness of this inverse source problem by reducing the problem to an operator equation for the source function.

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**Keywords:** heat equation, direct problem, inverse source problem, well-posedness, positive operator, Caputo fractional derivative.

**2020 Mathematics Subject Classification:** 35R11, 35R30

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# Optimal Difference Formulas in the Sobolev Spaces

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In the world today, the finite difference method is used for the numerical solution of many problems in physics and technology described by the equations of mathematical physics. The basic concepts of difference methods are approximation, stability, convergence, which are illustrated with examples of difference schemes for ordinary differential equations. For large numerical calculations, it becomes useful to optimize the process of approximate solution of ordinary differential equations, and if calculations are performed using difference formulas, then the formulas themselves are optimized. The problem of optimizing difference formulas in the modern understanding looks like the problem of finding the minimum norm of the error functional of the difference formula. Therefore, constructing optimal difference formulas in Sobolev spaces and finding the norm of the error functional of optimal difference formulas are urgent problems in computational mathematics.

This work is devoted to the study of optimal difference formulas in Sobolev space. There are algebraic and functional formulations of the problem of constructing difference formulas. In this work we consider functional formulations of the problem.

The functional formulation considers functions  $\varphi(x)$ , belonging to the Sobolev space  $L_2^{(m)}(0, 1)$ , where  $L_2^{(m)}$  is a Hilbert space whose elements are classes of real-valued functions differing on a polynomial of degree  $(m - 1)$  and square integrable with a derivative of order  $m$  on the interval  $[0, 1]$ .

We consider approximate solution of the Cauchy problem for first order ordinary linear differential equations. For this we suggest the functional method of construction of difference formulas. The error of the difference formula is estimated from above by the norm of the error functional of this formula. To find in explicit form the norm of the error functional  $\ell$  of difference formulas, we use the extremal function of this functional. Next, we find the extremal function in the Sobolev space  $L_2^{(m)}(0, 1)$  for any  $m \geq 2$ .

By minimizing the squared norm of the error functional  $\ell$  by the coefficients  $C^{(1)}[\beta]$  of difference formulas, we obtain a Wiener–Hopf type system for finding optimal coefficients  $\tilde{C}^{(1)}[\beta]$  difference formulas and optimal polynomial of discrete argument  $\tilde{P}_{m-2}[\beta]$ . Here we prove the existence and uniqueness of a solution to a Wiener–Hopf type system. At the same time, this work developed an algorithm for constructing optimal difference formulas in the Sobolev space  $L_2^{(m)}(0, 1)$  for any  $m \geq 2$ , with the help of which representations of the coefficients of optimal Adams-type difference formulas are obtained.

**Keywords:** optimal difference formulas, the error functional, Sobolev space, ordinary differential equations, approximation, stability, convergence.

**2020 Mathematics Subject Classification:** 65L05, 34A12

# Cylindrical Sobolev-Hardy Type Inequalities and Identities

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Let's revisit the Badiale-Tarantello conjecture [1]. Suppose we have  $x = (x', x'') \in \mathbb{R}^N \times \mathbb{R}^{n-N}$ , where  $2 \leq N \leq n$  and  $1 \leq p < N$ . In this context, we have an inequality represented as:

$$\left\| \frac{f}{|x'|} \right\|_{L^p(\mathbb{R}^n)} \leq C_{N,p} \|\nabla f\|_{L^p(\mathbb{R}^n)} \quad (1)$$

The constant  $C_{N,p} = \frac{p}{N-p}$  in this inequality was conjectured as sharp constant. It's worth noting that this conjecture, was initially established as true in [2] when  $p < N$ , and more recently, a different approach in [3] provided a proof. Additionally, by substituting the gradient with the Euler operator, [4] derived improved versions of this inequality, known as Sobolev type inequalities.

In this presentation, our primary focus will be on the critical case where  $p = N$  in the inequality (1). This specific case incorporates a logarithmic weight and an optimal constant. We will show several types of critical cases. Moreover, we'll delve into stability results and explore higher-order variations of the inequality. Interestingly, in the higher-order scenarios, we discovered an  $L^2$  identity that involves the well-known Oblong numbers and Stirling numbers of the second kind.

Lastly, we will demonstrate the application of these findings, showing Caffarelli-Kohn-Nirenberg-type inequalities with remainder terms that incorporate a logarithmic-type function as a weight.

This presentation is based on collaborative work with Nurgissa Yessirkegenov.

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**Keywords:** Cylindrical inequalities, critical Sobolev-Hardy type inequality, Stirling number.

**2020 Mathematics Subject Classification:** 22E30, 11B73, 26D10

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## Inverse Problem for Fractional Order Subdiffusion Equation

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We study the inverse problem of finding functions  $\{u(t), f\}$  that satisfy the following subdiffusion equation

$$D_t^\rho u(t) + Au(t) = g(t)f, \quad \rho \in (0, 1], \quad t \in (0, T], \tag{1}$$

with the initial

$$u(0) = \varphi, \tag{2}$$

and the additional conditions

$$\int_0^T u(t)dt = \psi. \tag{3}$$

Here  $g(t) \in C[0, T]$  is a given function and  $\varphi, \psi \in H$  are known elements,  $A$  is an unbounded positive self-adjoint operator and  $D_t^\rho$  stands for the Caputo fractional derivative.

We introduce the power of operator  $A$  with domain

$$D(A) = \{h \in H : \sum_{k=1}^\infty \lambda_k^2 |h_k|^2 < \infty\},$$

acting in  $H$  according to the rule:

$$Ah = \sum_{k=1}^\infty \lambda_k h_k v_k.$$

For  $0 < \rho < 1$  and an arbitrary complex number  $\mu$ , let  $E_{\rho, \mu}(z)$  denote the Mittag-Leffler function with two parameters of the complex argument  $z$  (see [1]):

$$E_{\rho, \mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + \mu)}.$$

**Lemma 2.** *Let  $\rho \in (0, 1]$ ,  $g(t) \in C^1[0, T]$  and  $g(0) \neq 0$ . Then there exist numbers  $m_0 > 0$  and  $k_0$  such that, for all  $T \leq m_0$  and  $k \geq k_0$ , the following estimates hold:*

$$\frac{C_0}{\lambda_k} \leq |p_{k, \rho}(T)| \leq \frac{C_1}{\lambda_k},$$

where

$$p_{k, \rho}(T) = \int_0^T g(\eta)(T - \eta)^\rho E_{\rho, \rho+1}(-\lambda_k(T - \eta)^\rho) d\eta$$

and constants  $C_0$  and  $C_1 > 0$  depend on  $m_0$  and  $k_0$ .

Let  $\mathbb{N} = K_\rho \cup K_{0, \rho}$ , where  $\mathbb{N}$  is the set of all natural numbers.  $K_\rho$  and  $K_{0, \rho}$  are sets such that: if  $p_{k, \rho}(T) \neq 0$ ,  $k \in K_\rho$ , otherwise, if  $p_{k, \rho}(T) = 0$ , then  $k \in K_{0, \rho}$ .

**Theorem 1.** *Let  $\rho \in (0, 1]$ ,  $\varphi \in H$ ,  $\psi \in D(A)$ ,  $g(t) \in C[0, T]$  and  $g(t) \neq 0$ ,  $t \in [0, T]$ . Then there exists a unique solution of the inverse problem (1)-(3).*

**Theorem 2.** *Let  $\rho \in (0, 1]$ ,  $\varphi \in H$ ,  $\psi \in D(A)$ ,  $g(t) \in C^1[0, T]$ . Further, we will assume that the conditions of Lemma 2 are satisfied and  $T$  is sufficiently small. If set  $K_{0, \rho}$  is empty, for all  $k$ , then there exists a unique solution of the inverse problem (1)-(3):*

$$f = \sum_{k=1}^\infty \frac{1}{p_{k, \rho}(T)} [\psi_k - \varphi_k T E_{\rho, 2}(-\lambda_k T^\rho)] v_k,$$

$$u(t) = \sum_{k=1}^{\infty} \left[ \varphi_k E_{\rho,1}(-\lambda_k t^\rho) + \frac{p_{k,\rho}(t)}{p_{k,\rho}(T)} [\psi_k - \varphi_k TE_{\rho,2}(-\lambda_k T^\rho)] \right] v_k.$$

If set  $K_{0,\rho}$  is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions

$$\psi_k = \varphi_k TE_{\rho,2}(-\lambda_k T^\rho), \quad k \in K_{0,\rho}$$

be satisfied. In this case, the solution to the problem (1)-(3) exists, but is not unique:

$$f = \sum_{k \in B_\rho} \frac{1}{p_k(T)} [\psi_k - \varphi_k TE_{\rho,2}(-\lambda_k T^\rho)] v_k + \sum_{k \in K_{0,\rho}} f_k v_k,$$
$$u(t) = \sum_{k=1}^{\infty} [\varphi_k E_{\rho,1}(-\lambda_k t^\rho) + f_k] v_k,$$

where  $f_k, k \in K_{0,\rho}$ , are arbitrary real numbers.

**Keywords:** subdiffusion equation, inverse problem, the Caputo derivative, Fourier method.

**2020 Mathematics Subject Classification:** Primary 35R11; Secondary 34A12.

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**Quality properties of multipoint problem for Schrodinger equations and applications**

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The existence, uniqueness, regularity properties and Strichartz type estimates for the solution of multipoint Cauchy problem for linear and nonlinear Schrödinger equations with general elliptic leading part are obtained.

**Key Word:** Schrödinger equations, elliptic operators, local solutions, Strichartz type inequalities, regularity properties of PDE

Consider the multipoint Cauchy problem for nonlinear Schrödinger equations (NLS)

$$(1.1) \quad i\partial_t u + Lu + F(u) = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, T],$$

$$(1.2) \quad u(0, x) = \varphi(x) + \sum_{k=1}^m \alpha_k u(\lambda_k, x), \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where  $L$  is an elliptic operator defined by

$$(1.3) \quad Lu = \sum_{|\beta| \leq 2l} a_\beta D^\beta u,$$

for  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ ,  $a_\beta \in \mathbb{C}$ ,  $m$  is an integer,  $\lambda_k \in (0, T]$ ,  $\alpha_k$  are complex numbers,  $F$  is a nonlinear operator and  $u = u(t, x)$  is an unknown function. If  $F(u) = \lambda |u|^p u$  in (1.1) we get the multipoint Cauchy problem for the nonlinear equation

$$(1.4) \quad i\partial_t u + Lu + \lambda |u|^p u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, T],$$

$$u(0, x) = \varphi(x) + \sum_{k=1}^m \alpha_k u(\lambda_k, x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where  $p \in (1, \infty)$ ,  $\lambda$  is a real number.

The existence of solutions and regularity properties of Cauchy problem for NLS equations studied e.g in [1 – 5], [7], [8] and the references therein. In contrast, to the mentioned above results we will study the existence, uniqueness and the regularity properties of the multipoint Cauchy problem (1.1) – (1.2).

**2. Definitions and background**

Let  $L_t^q L_x^r((a, b) \times \Omega)$  denotes the space of strongly measurable functions that are defined on the measurable set  $(a, b) \times \Omega$  with the norm

$$\|f\|_{L_t^q L_x^r((a,b) \times \Omega)} = \left( \int_a^b \left( \int_\Omega |f(t, x)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}, \quad 1 \leq q, r < \infty.$$

Let  $F$  denotes the Fourier transformation,  $\hat{u} = Fu$  and

$$s \in \mathbb{R}, \quad \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n, \quad |\xi|^2 = \sum_{k=1}^n \xi_k^2.$$



$S = S(\mathbb{R}^n)$  denotes the Schwartz class, i.e. the space of all complex-valued rapidly decreasing smooth functions on  $\mathbb{R}^n$  equipped with its usual topology generated by seminorms.  $S(\mathbb{R}^n)$  denoted by just  $S$ . Let  $S'(\mathbb{R}^n)$  denote the space of all continuous linear operators,  $L : S \rightarrow \mathbb{C}$ , equipped with the bounded convergence topology. Recall  $S(\mathbb{R}^n)$  is norm dense in  $L^p(\mathbb{R}^n)$  when  $1 < p < \infty$ . Let  $D'(\Omega)$  denote the class of generalized functions on  $\Omega \subset \mathbb{R}^n$ . Consider Sobolev space  $W^{s,p}(\mathbb{R}^n)$  and homogeneous Sobolev spaces  $\dot{W}^{s,p}(\mathbb{R}^n)$  defined by respectively,

$$W^{s,p}(\mathbb{R}^n) = \left\{ u : u \in S'(\mathbb{R}^n), \right. \\ \left. \|u\|_{W^{s,p}(\mathbb{R}^n)} = \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\},$$

$$\dot{W}^{s,p}(\mathbb{R}^n) = \left\{ u : u \in S'(\mathbb{R}^n), \|u\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \left\| F^{-1} |\xi|^s \hat{u} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\}.$$

Sometimes we use one and the same symbol  $C$  without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say  $\eta$ , we write  $C_\eta$ .

Let  $L$  be a differential operator defined by (1.3) and  $A = [a_{ij}]$ ,  $i, j = 1, 2, \dots, n$ .

**Condition 2.1.** Let  $L_0(\xi) =_{|\beta|=2l} a_\beta \xi^\beta \neq 0$  for  $\xi \in \mathbb{R}^n$ ,  $\xi \neq 0$  and there exists a positive constant  $M_0$  such that

$$|L(\xi)| = \left|_{|\beta| \leq 2l} a_\beta (i\xi)^\beta \right| \geq M_0 \left( |\xi|^{2l} + 1 \right)$$

for  $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$  with  $\xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2} \dots \xi_n^{\beta_n}$ .

**Definition 2.2.** Consider the initial value problem (1.1) – (1.2) for  $\varphi \in \dot{W}^{s,p}(\mathbb{R}^n)$ . This problem is critical when  $s = s_c := \frac{n}{2} - \frac{2}{p}$ , subcritical when  $s > s_c$ , and supercritical when  $s < s_c$ .

We write  $a \lesssim b$  to indicate that  $a \leq Cb$  for some constant  $C$ , which is permitted to depend on some parameters.

### 3. Dispersive and Strichartz type inequalities for linear Schrödinger equation

Assume the Condition 2.1 holds. The fundamental solution of the Schrödinger operator  $i\partial_t u + Lu$  is found as solution of the equation

$$(3.1) \quad i\partial_t u + Lu = \delta(t, x),$$

where  $\delta(t, x)$  is the generalized delta function. By applying the Fourier transformation with respect to  $x$  in (3.1) we get

$$(3.2) \quad i \frac{d}{dt} \hat{u}(t, \xi) - L(\xi) \hat{u}(t, \xi) = 1(\xi) \times \delta(t).$$

By using the Fourier transformation and by following [9, § 6.6,14.4] it can be shown that  $\hat{u}(t, \xi) = \theta(t) e^{-itL(\xi)}$ , i.e. the fundamental solution (3.1) can be expressed as

$$U(t, \cdot) = U_L(t, x) = F_x^{-1} \left[ \theta(t) e^{-itL(\xi)} \right],$$

where  $F_x$  is the Fourier transformation with respect to  $x$  and  $\theta(t)$  is the Heaviside unit function.

**Condition 3.1.** Assume  $n \geq 1$ ,

$$(3.3) \quad \frac{2}{q} + \frac{n}{r} \leq \frac{n}{2}, \quad 2 \leq q, r \leq \infty \quad \text{and} \quad (n, q, r) \neq (2, 2, \infty).$$

Assume  $H$  is an abstract Hilbert space and  $Q$  is a Hilbert space of function. Suppose for each  $t \in \mathbb{R}$  an operator  $U(t) : Q \rightarrow L^2(\Omega)$  obeys the following estimates:

$$(3.4) \quad \|U(t)f\|_{L^2_x(\Omega)} \lesssim \|f\|_H$$

for all  $t, \Omega \subset \mathbb{R}^n$  and all  $f \in Q$ ;

$$(3.5) \quad \|U(s)U^*(t)g\|_{L^\infty_x(\Omega)} \lesssim |t-s|^{-\frac{n}{2}} \|g\|_{L^1_x(\Omega)},$$

$$(3.6) \quad \|U(s)U^*(t)g\|_{L^\infty_x(\Omega)} \lesssim \left(1 + |t-s|^{-\frac{n}{2}}\right) \|g\|_{L^1_x(\Omega)}$$

for all  $t \neq s$  and all  $g \in L^1_x(\Omega)$ .

Consider the multipoint Cauchy problem for forced Schrodinger equation

$$(3.7) \quad i\partial_t u + Lu = f, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

$$u(t_0, x) = \varphi(x) + \sum_{k=1}^m \alpha_k u(\lambda_k, x), \quad x \in \mathbb{R}^n, \quad t_0, \lambda_k \in [0, T], \quad \lambda_k > t_0.$$

We are now ready to state the standard Strichartz estimates:

**Lemma 3.3.** Assume the Condition 2.1 is satisfied,  $\varphi \in \tilde{W}^{\gamma,p}(\mathbb{R}^n)$  for  $\gamma \geq \frac{n}{p}$  and  $p \in [1, \infty]$ . Then problem (3.7) has a unique generalized solution.

**Proof.** By using the Fourier transform we get from (3.7) :

$$(3.8) \quad i\hat{u}_t(t, \xi) - L(\xi)\hat{u}(t, \xi) = \hat{f}(t, \xi),$$

$$\hat{u}(0, \xi) = \hat{\varphi}(\xi) + \sum_{k=1}^m \alpha_k \hat{u}(\lambda_k, \xi), \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

where  $\hat{u}(t, \xi)$  is a Fourier transform of  $u(t, x)$  with respect to  $x$ .

Consider the problem

$$(3.9) \quad \hat{u}_t(t, \xi) + iL(\xi)\hat{u}(t, \xi) = \hat{f}(t, \xi),$$

$$\hat{u}(0, \xi) = u_0(\xi), \quad \xi \in \mathbb{R}^n, \quad t \in [0, T],$$

where  $u_0(\xi) \in \mathbb{C}$  for  $\xi \in \mathbb{R}^n$ . In view of Condition 2.1, by [6, §11]  $-iL(\xi)$  is a generator of a strongly continuous  $C_0$  groups  $U_L(t, \xi) = e^{-itL(\xi)}$ . Hence, the Cauchy problem (3.9) has a solution for all  $\xi \in \mathbb{R}^n$  and the solution can be expressed as

$$(3.10) \quad \hat{u}(t, \xi) = e^{-itL(\xi)}u_0(\xi) + \int_{t_0}^t e^{-itL(\xi)|t-\tau|} \hat{f}(\tau, \xi) d\tau, \quad t \in (0, T).$$

Using the formulas (3.8) – (3.10), we get that the solution of (3.7) will be expressed as the following formula:

$$u(t, x) = V(t)\varphi(x) + \sum_{k=1}^m \alpha_k V_k(t, x) + \sum_{k=1}^m \alpha_k G_k(t, x) + G_0(t, x),$$

where

$$(3.11) \quad V(t) = F^{-1}[U_L(t, \xi)\hat{\varphi}(\xi)], \quad V_k(t, x) = F^{-1}[U_L(\lambda_k, \xi)\hat{\varphi}(\xi)],$$

$$G_k(t, x) = F^{-1} \left[ \int_{t_0}^{\lambda_k} U_L(\lambda_k - \tau, \xi) \hat{f}(\tau, \xi) d\tau \right],$$

$$G_0(t, x) = F^{-1} \left[ \int_{t_0}^t U_L(t - \tau, \xi) \hat{f}(\tau, \xi) d\tau \right].$$

By following [7, Theorem 1.2] we have:

**Theorem 3.1.** Assume  $U(t)$  obeys (3.5) and (3.6). Then the following estimates are hold

$$(3.12) \quad \|U(t) f\|_{L_t^q L_x^r} \lesssim \|f\|_H,$$

$$(3.13) \quad \|U^*(s) f(s) ds\|_Q \lesssim \|f\|_{L_t^{q'} L_x^{r'}},$$

$$(3.14) \quad_{s < t} \|U(t) U^*(s) f(s) ds\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

for all pairs  $(q, r), (\tilde{q}, \tilde{r})$ .

**Proof.** By duality, (3.13) is equivalent to (3.14). By the  $TT^*$  method, (3.14) is in turn equivalent to the bilinear form estimate

$$(3.15) \quad |\langle U^*(s) f(s), U^*(t) G(t) \rangle ds dt| \lesssim \|f\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

By symmetry it suffices to show the retarded version of (3.13)

$$(3.16) \quad |T(f, G)| \lesssim \|f\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

where  $T(F, G)$  is the bilinear form defined by

$$T(F, G) = \int_{s < t} \langle U(s)^* f(s), (U(t))^* G(t) \rangle ds dt.$$

Then by reasoning as in [10, Theorem 3.1], by using the estimates (3.15), (3.16) and the Lemma 3.3, we get the conclusion.

**Theorem 3.2.** Assume the Conditions 2.1 and 3.1 are satisfied. Let  $0 \leq s \leq 1, \varphi \in \dot{W}^{s,2}(\mathbb{R}^n), f \in N^0([0, T]; \dot{W}^{s,2}(\mathbb{R}^n))$  and let  $u : \sigma \rightarrow \mathbb{C}$  be a solution to (3.7). Then

$$(3.17) \quad \|\nabla|^s u\|_{S^0([0, T])} + \|\nabla|^s u\|_{C^0([0, T]; L^2(\mathbb{R}^n))} \lesssim$$

$$\|\nabla|^s \varphi\|_{L^2(\mathbb{R}^n)} + \|\nabla|^s f\|_{N^0([0, T])}.$$

**Proof.** Let  $2 \leq q, r, \tilde{q}, \tilde{r} \leq \infty$  with

$$\frac{2}{q} + \frac{n}{r} = \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2}.$$

If  $n = 2$ , we also require that  $q, \tilde{q} > 2$ . Consider first, the nonendpoint case. By Lemma 3.3 the problem has a solution. The linear operators in (3.12)-(3.14) are adjoint of one another; thus, by the method of  $TT^*$  both will follow once we prove

$$\left\| \int_{s < t} U_L(t-s, \cdot) f(s) ds \right\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^{q'} L_x^{r'}}.$$

Apply Theorem 3.1 with  $Q = L_x^2(\mathbb{R}^n) = L_x^2$ . The energy estimate (3.12):

$$\|U_L(t, \cdot) f\|_{L_x^2} \lesssim \|f\|_{L_x^2}$$

follows from Plancherel's theorem, the untruncated decay estimate

$$\|U_L(t-s, \cdot) f\|_{L_x^\infty} \lesssim |t-s|^{-\frac{n}{2}} \|f\|_{L_x^1},$$

and explicit representation of the Schrodinger evolution operator  $U_L(t) = U_L(t, x)$ . Due to properties gropes  $U_L(t)$  and by the dispersive estimate (3.4) we have

$$|\Phi| \lesssim_{s < t} |U_L(t-s) ds|_{B(H)} |f(s)| ds \lesssim_{\mathbb{R}} |t-s|^{-n(\frac{1}{2} - \frac{1}{p})} |f(s)| ds,$$

where

$$\Phi = \int_{s < t} U_L(t-s) f(s) ds.$$

Then by reasoning as in [10, Theorem 3.2], we obtain the conclusion.

**4. Strichartz type estimates for solution of the nonlinear Schrödinger equation**

Consider the multipoint initial-value problem (1.1) – (1.2).

**Condition 4.1.** Assume that the function  $F : \mathbb{C} \rightarrow \mathbb{C}$  is continuously differentiable and obeys the power type estimates

$$(4.1) \quad F(u) = O(|u|^{1+p}), \quad F_u(u) = O(|u|^p),$$

$$(4.2) \quad F_u(v) - F_u(w) = O(|v - w|^{\min\{p,1\}} + |w|^{\max\{0,p-1\}})$$

for some  $p > 0$ , where  $F_u(u)$  denotes the derivative of operator function  $F$  with respect to  $u$ .

From (4.1) we obtain

$$(4.3) \quad |F(u) - F(v)| \lesssim |u - v|(|u|^p + |v|^p).$$

**Remark 4.1.** The model example of a nonlinearity obeying the conditions above is  $F(u) = |u|^p u$ ,  $p \in (1, \infty)$  for which the critical homogeneous Sobolev space is  $\dot{W}_x^{s_c,2}(\mathbb{R}^n)$  with  $s_c := \frac{n}{2} - \frac{2}{p}$ .

**Definition 4.1.** A function  $F : \sigma \rightarrow \mathbb{C}$  is called a (strong) solution to (1.1) – (1.2) if it lies in the class

$$C_t^0([0, T]; \dot{W}_x^{s_c,2}(\mathbb{R}^n)) \cap L_t^{p+2} L_x^{\frac{np(p+2)}{4}}(\sigma)$$

and obey:

$$(4.4) \quad u(t, x) = V(t) \varphi(x) + \sum_{k=1}^m \alpha_k V_k(t, x) + \sum_{k=1}^m \alpha_k G_k(t, x) + G_0(t, x),$$

where

$$(4.5) \quad \begin{aligned} V(t) &= F^{-1}[U_L(t, \xi) \hat{\varphi}(\xi)], \quad V_k(t, x) = F^{-1}[U_L(\lambda_k, \xi) \hat{\varphi}(\xi)], \\ G_k(t, x) &= F^{-1} \left[ \int_{t_0}^{\lambda_k} U_L(\lambda_k - \tau, \xi) \hat{F}(\tau, \xi) d\tau \right], \\ G_0(t, x) &= F^{-1} \left[ \int_{t_0}^t U_L(t - \tau, \xi) \hat{F}(\tau, \xi) d\tau \right]. \end{aligned}$$

We say that  $u$  is a global solution if  $T = \infty$ .

**Theorem 4.1.** Assume the Conditions 2.1, 3.1, 4.1 are satisfied. Let  $0 \leq s \leq 1$ ,  $\varphi \in \dot{W}_x^{s,2}(\mathbb{R}^n)$  and  $n \geq 1$ . Then there exists  $\eta_0 = \eta_0(n) > 0$  such that if  $0 < \eta \leq \eta_0$  such that

$$\| |\nabla|^s U_L(t) \varphi \|_{L_t^{p+2} L_x^{\frac{\sigma}{p}}(\sigma)} \leq \eta,$$

then here exists a unique solution  $u$  to (4.1) on  $[0, T] \times \mathbb{R}^n$ . Moreover, the following estimates hold

$$\begin{aligned} \| |\nabla|^s U_L u \|_{L_t^{p+2} L_x^{\frac{\sigma}{p}}(\sigma)} &\leq 2\eta, \\ \| |\nabla|^s u \|_{S^0(\sigma)} + \| u \|_{C^0([0,T]; \dot{W}_x^{s,2}(\mathbb{R}^n))} &\lesssim \| |\nabla|^s \varphi \|_{L_x^2(\mathbb{R}^n)} + \eta^{1+p}, \\ \| u \|_{S^0(\sigma; H)} &\lesssim \| \varphi \|_{L_x^2(\mathbb{R}^n; H)}, \quad r = r(p, n) = \frac{2n(p+2)}{2(n-2) + np}. \end{aligned}$$

**Proof.** We apply the standard fixed point argument. More precisely, using the estimates (3.26), the equalitie (3.11) and by reasoning as in [10, Theorem 4.1], we obtain the conclusion.

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# Time Nonlocal Schrödinger Equation with Spatial Periodic Boundary Condition in Fractional Power Hilbert Spaces

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In this study, time nonlocal Schrödinger equation with spatial periodic boundary condition in fractional power Hilbert spaces is investigated. Stable first order of accuracy Rothe difference scheme and second order of accuracy Crank-Nicholson difference scheme for the numerical solution of this problem is presented. The main theorem on the stability of these difference schemes are established. Numerical results are given.

**Keywords:** difference schemes, stability, Schrödinger equation

**2020 Mathematics Subject Classification:** 35J25, 47E05, 34B27

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# Numerical Algorithms for Solving the Inverse Problem of Identifying the Initial Conditions of a Subdiffusion Equation

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The paper considers the parallel algorithm for solving the retrospective inverse problem of identifying the initial value for a subdiffusion equation. To solve this inverse problem, the regularized iterative conjugate gradient method is used. At each iteration of the method, we need to solve the auxiliary direct initial-value problem. Using the finite difference scheme, the direct problem is reduced to a large number of systems of linear algebraic equations.

The following subdiffusion equation is considered:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} - \mathcal{L}u &= f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, 0 < t \leq T, \\ u(\mathbf{x}, t) &= 0, \quad \mathbf{x} \in \partial\Omega, 0 < t \leq T, \\ u(\mathbf{x}, 0) &= \varphi(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned}$$

Here,  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \bar{\Omega} = \prod_{i=1}^n [a_i, b_i]$ ,  $0 < \alpha < 1$ , and  $\mathcal{L}$  is an elliptic operator

$$\mathcal{L}u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( k_i(\mathbf{x}, t) \frac{\partial u}{\partial x_i} \right), \quad x_i \in (a_i, b_i), 0 < t \leq T.$$

The fractional Caputo derivative with order  $\alpha$  is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} (t-s)^{-\alpha}, \quad \mathbf{x} \in \Omega, 0 < t \leq T.$$

In the present work, we study the inverse problem of restoring the initial condition  $\varphi(\mathbf{x})$ . Additional information on the solution is given in the form

$$u(\mathbf{x}, t) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

For the numerical solution of the inverse problem, the iterative conjugate gradient method [1] is used, while at each iteration the direct problem is solved by using an implicit difference scheme [2]. The direct problem is reduced to solving a large system of linear algebraic equation with triadiagonal (for the 1D case) or block triadiagonal (for the 2D and larger dimensions) matrix at each time step.

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**Keywords:** fractional differential equations, inverse problems, retrospective problem.

**2020 Mathematics Subject Classification:** 35R30, 65R32

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# Maximizing Schatten p-Norms and Related Isoperimetric Inequalities

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In this presentation, we explore the intriguing realm of Schatten p-norms and their relationship with Riesz potential operators within domains of fixed measure. Our main result unveils the ball as an optimal maximizer for these integer order Schatten p-norms. This finding extends to the polyharmonic Newton potential operator, a vital component in nonlocal boundary value problems associated with the poly-Laplacian. This extension mirrors the pioneering work of M. Kac and T. Kalmenov, originally established for the Laplacian, leading us to the derivation of isoperimetric inequalities for the eigenvalues. These inequalities, akin to the classical Rayleigh–Faber–Krahn and Hong–Krahn–Szego counterparts, further illuminate the structural properties of the polyharmonic Newton potential operator.

In our presentation, we delve into the broader landscape by considering extensions of these results to convolution-type integral operators, offering explicit examples to elucidate our findings. Additionally, we present a new insight into the multidimensional MEMS (micro-electro mechanical systems) problem within the Euclidean space  $R^d$ , where  $d \geq 3$ . Our investigation reveals that minimizing the pull-in voltage for this problem involves the symmetrization of the permittivity profile, and we establish the foundation for this claim through the application of Talenti’s comparison principle. This talk is based on our joint works [1]-[3].

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# Critical Exponents for the Exterior Evolution Problems

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The paper studies the large-time behavior of solutions to the Robin problem for PDEs with critical nonlinearities. For the considered problems, nonexistence results are obtained, which complements the interesting recent results by Ikeda et al. [1], where critical cases were left open. Moreover, our results provide partially answers to some other open questions previously posed by Zhang [2] and Jleli-Samet [3]. The proof of main results is based on methods of nonlinear capacity estimates specifically adapted to the nature of the exterior domain. Furthermore, the difference in our approach lies in the fact that we are considering a class of test functions with logarithmic arguments.

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**Keywords:** exterior Robin problem, critical exponent, nonexistence, global solution.

**2020 Mathematics Subject Classification:** 35K70, 35A01, 35B44

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# **On Fourier Multiplier on Non-Commutative Torus and Applications**

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In this work, we deal with the Fourier multiplier on non-commutative torus and its applications in non-commutative Harmonic analysis.

# On Solvability of Some Inverse Problems for a Nonlocal Fourth-Order Parabolic Equation with Multiple Involution

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In this paper, the solvability of some inverse problems for a nonlocal analogue of a fourth-order parabolic equation is studied. For this purpose, a nonlocal analogue of the biharmonic operator is introduced. When defining this operator, transformations of the involution type are used. In a parallelepiped, the eigenfunctions and eigenvalues of the Dirichlet type problem for a nonlocal biharmonic operator are studied. The eigenfunctions and eigenvalues for this problem are constructed explicitly and the completeness of the system of eigenfunctions is proved. Two types of inverse problems on finding a solution to the equation and its right-hand side are studied. In the first problem, the right-hand side depending on the spatial variable is sought, and in the second problem, a function depending on the time variable is found. The first problem is solved by using the Fourier variable separation method, and the second problem by reducing it to solving an integral equation. The theorems on the existence and uniqueness of the solution are proved.

Note that similar problems in the case  $n = 2$  of a classical parabolic equation in a rectangular domain were studied in [1,2].

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**Keywords:** inverse problem, nonlocal biharmonic operator, parabolic equation, eigenfunction, eigenvalue, Fourier method, existence of solution, uniqueness of solution.

**2020 Mathematics Subject Classification:** 34K06, 34K08, 35G16

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# Direct and Inverse Problems for the Burgers Equation in Degenerating Domains

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Let consider the following nonlinearly degenerating domain

$$\Omega = \{x, t \mid \varphi_1(t) < x < \varphi_2(t), 0 < t < T < \infty\},$$

with its cross section  $\Omega_t = \{\varphi_1(t) < x < \varphi_2(t)\}$  for a fixed value of the time variable  $t \in (0, T)$ , with

$$\varphi_1(0) = \varphi_2(0).$$

The functions  $\varphi_1(t)$  and  $\varphi_2(t)$  are defined on  $[0, T]$ , and belong to  $C^1(0, T)$ .

In the domain  $\Omega$  consider the following inverse problem for the Burgers equation

$$\partial_t u(x, t) + u(x, t)\partial_x u(x, t) - \nu \partial_x^2 u(x, t) = w(t)f_t(x), \text{ in } \Omega, \tag{1}$$

$$\partial_x^j u(\varphi_1(t), t) = \partial_x^j u(\varphi_2(t), t), \quad j = 0, 1; t \in (0, T), \tag{2}$$

$$u(0, 0) = 0, \tag{3}$$

$$\int_{\varphi_1(t)}^{\varphi_2(t)} u(x, t)dx = E(t), \quad t \in (0, T), \tag{4}$$

where  $\nu = const > 0$  is a given constant and functions  $f_t(x)$ ,  $E(t)$  satisfy the conditions

$$\begin{cases} f_t(x) \equiv f(x, t) \in L^\infty(0, T; L^\infty(\Omega_t)), E(t) \in W^{1,\infty}(0, T), E(0) = 0, \\ \tilde{f}(t) \equiv \int_{\varphi_1(t)}^{\varphi_2(t)} f_t(x)dx \neq 0, \forall t \in (t_0, T), t_0 > 0, \tilde{f}(t) = O(\varphi(t)), t \rightarrow 0+, \end{cases} \tag{5}$$

here  $\varphi(t) = \varphi_2(t) - \varphi_1(t)$ .

The work will study the question of solvability of the inverse problem, namely, what conditions must be satisfied by the functions along which the domain changes in order for the inverse problem to be uniquely solvable. This work is a logical continuation of works [1,2].

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**Keywords:** Inverse problem, Burgers equation, non-parabolic domain, periodic boundary condition, integral overdetermination condition.

**2020 Mathematics Subject Classification:** 35K55, 35K10, 35K20

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# Hardy and Rellich Type Identities and Inequalities Related to Baouendi-Grushin Operator

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Recall from [1] that the Hardy inequality for Baouendi-Grushin vector fields takes the form

$$\int_{\mathbb{R}^n} (|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2) dz \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} \left(\frac{|x|^{2\gamma}}{|x|^{2+2\gamma} + (1+\gamma)^2 |y|^2}\right) |f|^2 dz, \quad (1)$$

where  $z = (x_1, \dots, x_m, y_1, \dots, y_k) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$  with  $n = m + k$ ,  $m, k \geq 1$ ,  $\gamma \geq 0$ ,  $Q = m + (1 + \gamma)k$  and  $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k \setminus \{(0, 0)\})$ . Here,  $\nabla_x f$  and  $\nabla_y f$  are the gradients of  $f$  in the variables  $x$  and  $y$ , respectively.

In this talk, we discuss sharp remainder formulae for Hardy and Rellich inequalities related to the Baouendi-Grushin operator involving radial derivatives in some of the variables, which improve the classical Hardy and Rellich inequalities as well as (1). If time permits, we also discuss magnetic functional inequalities related to the Baouendi-Grushin operator with Aharonov-Bohm type magnetic field.

This talk is based on the joint research with Ari Laptev (Imperial College London, UK) and Michael Ruzhansky (Ghent University, Belgium) [2]-[3], and with Amir Zhangirbayev.

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**Keywords:** Hardy inequalities, Baouendi-Grushin operator, Aharonov-Bohm magnetic field.

**2010 Mathematics Subject Classification:** 26D10, 35P15

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# Iterated Discrete Hardy-Type Inequalities: the Case $\theta < P$

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Let  $0 < q, p, \theta < \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ . Let  $\varphi = \{\varphi_i\}_{i=1}^\infty$  be a sequence of non-negative numbers,  $u = \{u_i\}_{i=1}^\infty$  and  $w = \{w_i\}_{i=1}^\infty$  be sequences of positive numbers, which will be called the weight sequences. Let us denote by  $l_{p,u}$  the space of all sequences  $f = \{f_i\}_{i=1}^\infty$  of real numbers such that

$$\|f\|_{p,u} = \left( \sum_{i=1}^\infty |u_i f_i|^p \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

For any  $f \in l_{p,u}$  we characterize the following iterated discrete Hardy-type inequality with three weights

$$\left( \sum_{n=1}^\infty w_n^\theta \left( \sum_{k=1}^n \left| \varphi_k \sum_{i=1}^k f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \leq C \left( \sum_{i=1}^\infty |u_i f_i|^p \right)^{\frac{1}{p}}, \tag{1}$$

where  $C$  is a positive constant independent of  $f$ . The dual discrete version of inequality (1) has the form

$$\left( \sum_{n=1}^\infty w_n^\theta \left( \sum_{k=n}^\infty \left| \varphi_k \sum_{i=k}^\infty f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \leq C \left( \sum_{i=1}^\infty |u_i f_i|^p \right)^{\frac{1}{p}}. \tag{2}$$

Paper [1] also contains results for inequality (2) for the case  $0 < p \leq 1$ , but when  $p \leq \min\{q, \theta\} < \infty$ . In paper [2], discrete Hardy-type inequality (1) have been characterized for the same relations between  $p, \theta$  and  $q$ , namely, for the cases  $p \leq \theta < \infty, 0 < q$  and  $\theta < p < \infty, 0 < q < \theta$ . Here we consider the most difficult case  $\theta < p < \infty$  and  $0 < \theta < q$  or, equivalently,  $0 < \theta < \min\{p, q\} < \infty$ , which has no explicit

**Theorem 1.** *Let  $0 < \theta < \min\{p, q\} < \infty, p > 1$ . Then inequality (1) holds if and only if  $B_2 < \infty$ , where*

$$B_2 = \left[ \sum_{i=1}^\infty u_i^{-p'} \left( \sum_{j=1}^i u_j^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \left( \sum_{n=i}^\infty w_n^\theta \left( \sum_{k=i}^n \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover,  $C \approx B_2$ , where  $C$  is the best constant in (1).

**Theorem 2.** *Let  $0 < \theta < \min\{p, q\} < \infty, p > 1$ . Then inequality (2) holds if and only if  $B_1 < \infty$ , where*

$$B_1 = \left[ \sum_{i=1}^\infty u_i^{-p'} \left( \sum_{j=i}^\infty u_j^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \left( \sum_{n=1}^i w_n^\theta \left( \sum_{k=n}^i \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover,  $C \approx B_1$ , where  $C$  is the best constant in (2).

**Theorem 3.** *Let  $0 < \theta < \min\{p, q\} < \infty, 0 < p \leq 1$ . Then inequality (1) holds if and only if  $B_4 < \infty$ , where*

$$B_4 = \left[ \sum_{i=1}^\infty u_i^{-\frac{\theta p}{p-\theta}} \left( \sum_{n=i}^\infty w_n^\theta \left( \sum_{k=i}^n \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover,  $C \approx B_4$ , where  $C$  is the best constant in (1).

**Theorem 4.** Let  $0 < \theta < \min\{p, q\} < \infty$ ,  $0 < p \leq 1$ . Then inequality (2) holds if and only if  $B_3 < \infty$ , where

$$B_3 = \left[ \sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}} \left( \sum_{n=1}^i w_n^\theta \left( \sum_{k=n}^i \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover,  $C \approx B_3$ , where  $C$  is the best constant in (2).

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**Keywords:** discrete inequality, Hardy-type operator, weight sequence, sequence space, characterization.

**2020 Mathematics Subject Classification:** 26D15, 26D20

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## Geometry and Solutions of the Planar Problem of Two Centers of Gravitation

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**Abstract.** In 1760 Euler found a second “momentum - like” integral for the planar problem of two fixed centers of gravitation. Thus the motion of a free test particle under the influence of the two centers turned out to be completely integrable with two degrees of freedom. The solutions of Jacobi and Königsberger in terms of elliptic theta-functions contain complete Abelian integrals and that’s why they are not convenient for use.

We present effective solutions of the problem, classify and illustrate all the non-degenerated motions of the test particle. There exist exactly 16 topologically different types of motions, 10 of them are bonded and 6 are unbounded.



# Construction of Stable Control System for a Given Program Manifold

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We consider the problem of construction of control automatic systems by given  $(n-s)$ -dimensional program manifold  $\Omega(t) \equiv \omega(t, x) = 0$ , in the following form:

$$\dot{x}(t) = f(t, x) - B_1 \xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T \omega, \quad t \in I = [0, \infty), \quad (1)$$

where  $x \in R^n$  is a state vector of the object,  $f \in R^n$  is a vector-function, satisfying to conditions of existence of a solution  $x(t) = 0$ , and  $B_1 \in R^{n \times r}$ ,  $P \in R^{n \times r}$  are matrices,  $\omega \in R^s$  ( $s \leq n$ ) is a vector,  $\xi \in R^r$  is a differentiable on  $\sigma$  vector-function, satisfying of local quadratic connection

$$\begin{aligned} \varphi(0) = 0 \wedge 0 < \sigma^T \varphi(\sigma) \leq \sigma^T K \sigma, \quad \forall \sigma \neq 0, \\ K_1 \leq \frac{\partial \varphi(\sigma)}{\partial \sigma} \leq K_2, \{K = K^T > 0\} \in R^{r \times r}, K_i = K_i^T > 0. \end{aligned} \quad (2)$$

**Definition 1.** The program manifold  $\Omega(t)$  of automatic control system is called absolutely stable if it is globally stable on solutions of system (1) for any  $\omega(t_0, x_0)$  and  $\varphi(\sigma)$ , satisfying conditions (2).

Differentiating  $\omega$  from (2) by time  $t$  based on (1) we obtain

$$\dot{\omega} = \frac{\partial \omega}{\partial t} + Hf(t, x) - HB_1 \xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T \omega, \quad t \in I = [0, \infty), \quad (16)$$

Choosing the function  $F(t, x, \omega)$  in linear form with respect to  $\omega$  we obtain the following system:

$$\dot{\omega}(t) = A\omega - B\xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T \omega, \quad t \in I = [0, \infty).$$

Here nonlinearity satisfies to generalized conditions (15), and  $F(t, x, \omega) = -A\omega$ ,  $A \in R^{s \times s}$ ,  $H = \frac{\partial \omega}{\partial x}$ ,  $B = HB_1$ .

For system (3), we construct the Lyapunov function in the following form

$$V(\omega) = \omega^T L \omega, \quad (4)$$

where  $L$  is a positive-definite symmetric  $s \times s$  matrix. The derivative of the function  $V(\omega)$  in time  $t$  by virtue of the system (17) and taking into account the condition (15) will take the form

$$-\dot{V}(\omega) = \omega^T C \omega + \sigma^T K \xi, \quad (5)$$

if the following matrix equalities are true

$$A^T l + LA = C, \quad (6)$$

$$PK = 2LB. \quad (7)$$

**Theorem 1.** Let a definite-positive matrix  $C$  be given, the nonlinearity  $\varphi(\sigma)$  satisfies the conditions (2), there exists a matrix  $L$ . satisfying equation (6). Then for the absolute stability of the program manifold  $\Omega(t)$  with respect to a given vector function  $\omega$  it is enough to perform matrix equalities (6), (7).

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**Keywords:** Stability, program manifold, control systems, Lyapunov function.

**2020 Mathematics Subject Classification:** 34K20, 34K32, 93C15

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## The evolution of a Michaelis-Menten type dynamics of tum

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Here, we investigate Michaelis-Menten type dynamics of phase-space analysis to a mathematical model of tumor growth with an immune responses. We then explore the effects of adaptive cellular immunotherapy on the model and describe under what circumstances the tumor can be eliminated. The addition of a drug term to the system can move the solution trajectory into a desirable basin of attraction.

One of main aims is derivation of sufficient conditions under which the possible biologically feasible dynamics is local and globally stable, and a converges to one of equilibrium points. Since these equilibrium points have a biological sense, we notice that understanding limit properties of dynamics of cells populations based on solving problem nonlinear dynamical system may be of an essential interest for the prediction of health conditions of a patient without a treatment, when the data (e.g. the status of blood cells shown above) that determines the condition of the patient are compared at various times  $t_0, t_1, \dots, t_m$  and correlated.

Mathematical analysis of the Michaelis-Menten type equations, regarding to dissipativity, boundedness of solutions, nature of equilibria, local and global stability have been investigated.

We studied some features of behavior of one of three-dimensional tumor growth models with dynamics described in terms of densities of three cells populations: tumor cells, healthy host cells and effector immune cells.

**Keywords:** Cancer tumor model, Mathematical modeling, Immune system, Stability of dynamical systems, Equilibrium point, Multiphase attractors.

**2020 Mathematics Subject Classification:** 3M99, 34D20, 34D05, 37C75

Bahçeşehir University (Türkiye), Analysis & PDE Center, Ghent University (Belgium), and Institute of Mathematics and Mathematical Modeling (Kazakhstan), being the organizers of the conference, present this Abstract book, which contains brief abstracts of the reports of the participants of the International Mathematical Conference «Functional Analysis in Interdisciplinary Applications» (FAIA2023).

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